Quantum Mechanics of Collision Processes*
[Scattering?]

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Abstract

The Schrödinger form of Quantum Mechanics permits one to define in a natural way the frequency of occurrence of a state with help of the intensity of the associated eigenfunction. This understanding carries over to the theory of scattering where the transition probability is determined through the asymptotic cases of aperiodic solutions.

1 Introduction

Collision processes have not only supplied convincing experimental proof for the basic assumptions of Quantum Theory, but also seem suitable to explain more about the physical meaning of the formal laws of the so-called "Quantum Mechanics". On the one hand it seems that Quantum Mechanics always gives the correct values for stationary states and the correct values for radiative transitions. However, with regards to the physical interpretation of the formulas, opinions are divided. The authors of the matrix formulation of Quantum Mechanics[1] are of the opinion that an exact description of the processes in space and time are principally impossible. They are content therefore with a list of relations between physical observables. Only in the classical limit can these physical observables be interpretted as posessing the characteristics of [classical?] motion.

Schrödinger[2] on the other hand, appears to ascribe to the wave (which he, according to de Broglie's process, regards as the carrier of atomic processes) the same kind of characteristics as those which a light wave possesses. He tries to construct wave groups

*See the preliminary publication, ZS. f. Phys. **37**, 863, 1926.

[1]W. Heisenberg, ZS. f. Phys. **33**, 879, 1925; M Born and P Jordan, ibid. **34**, 858, 1925. See also P.A.M. Dirac, Proc, Roy, Soc, **109**, 642, 1925; **110**, 561, 1926.

[2]E. Schrödinger, Ann. d. Phys. **79**, 361, 489, 734, 1926. Vlg. besonders die zweite Mitteilung, S. 499. Ferner Naturw. **14**, 664, 1926.

[wave packets?] which have relatively small dimensions in all directions and should, as it seems [to Born], directly represent moving corpuscles.

Neither of these two views seem satisfactory to me. I would like to attempt here a third interpretation and test its applicability to collision processes. I thereby pin my hopes on a comment of Einstein's regarding the relationship between the wave field and light quanta. He says roughly that the waves may only be seen as guiding [showing?] the way for corpuscular light quanta, and he spoke in the same sense of a "ghost field". This determines the probability that one light quantum, which is the carrier of energy and momentum, chooses a particular [definite?] path. The field itself, however, does not have energy or momentum.

If one wants to bring Einstein's thoughts into direct relation with Quantum Mechanics, one is better off waiting until the electromagnetic field has been included into the formalism. With regards to the complete analogy between a light quantum and the electron, one should not forget to formulate the laws of electron motion in a similar manner. And here it is obvious to regard the de Broglie-Schrödinger waves as a "ghost field", or even better as a guiding field.

I would like to pursue the association further: The guiding field, represented by a scalar function ψ of the coördinates of all participating particles and time, evolves according to Schrödinger's differential equation. Energy and momentum, however, are transferred as if corpuscles (electrons) are literally flying around [behaving classically?]. The paths of these corpuscles are only determined insofar as energy and momentum conservation restrict them. Apart from that, we only have a probability for the choice of a certain path. This probability is given by the value of the distribution of the function ψ. One could summarise approximately, somewhat paradoxically: The movement of particles follows a probability law, the probability itself however evolves in accordance with the law of causality[3].

If one views overall the three steps of the development of Quantum Mechanics, one sees that the first, namely periodic processes, is wholly unsuitable for verifying the usefulness of [the proposed] picture. The second step, namely aperiodic stationary processes, is somewhat more productive. It is this subject which will occupy us in this article. Really the third step should be seen as the most significant one ie: non-stationary events. These processes must show whether the interference of damped probability waves is sufficient to explain phenomena which apparently point to a coupling independent of space and time.

A more precise description is only possible if grounded in a mathematical development[4]. It is to this development that we now turn; later we will return to the hypothesis itself.

[3] I define the law of causality thus: that the complete knowledge of a state at a certain instant determines the distribution of the state for all later times.

[4] Herr Prof. N. Wiener of Cambridge, Mass. has helped me with the mathematical aspects of this work in a most friendly way. For this, I would like to express my thanks and acknowledge that without him, I would have been unable to reach my goal.

2 Definition of Weights and Frequencies of occurrence for Periodic Systems

We begin with a completely formal view of a stationary non-degenerate state. This system may be characterised by Schrödinger's differential equation:

$$[H - W, \psi] = 0\,. \tag{1}$$

The eigenfunctions are normalised to unity[5]:

$$\int \psi_n(q)\psi_m^*(q)dq = \delta_{nm}\,. \tag{2}$$

Any arbitrary function $\psi(q)$ may be expanded in terms of eigenfunctions

$$\psi(q) = \sum_n c_n\psi_n(q)\,. \tag{3}$$

So far we have only directed our attention towards the eigenfuctions ψ_n and eigenvalues W_n. the picture developed in the introduction suggested that equation 3 should be related to the probability that in a collection of equal, uncoupled atoms, a state should occur with a certain frequency [multiplicity?]. The completeness relation

$$\int |\psi(q)|^2\, dq = \sum_n |c_n|^2 \tag{4}$$

suggests regarding this integral as representing the number of atoms. We have for a single normalised eigenfunction a value of 1. (Or a-priori, the state has weight 1.) $|c_n|^2$ represents the frequency of occurrence of a state n and the entire number of atoms is assembled additively out of these pieces.

In order to justify this interpretation, we consider for example the motion of a point mass in 3-dimensional space under the influence of a potential energy $U(x, y, z)$. Then the differential equation reads

$$\Delta\psi + \frac{2\mu}{\hbar^2}\,(W - U)\,\psi = 0\,. \tag{5}$$

One substitutes here for W and ψ, an eigenvalue W_n and eigenfunction ψ_n, multiply the equation by ψ_m^* and integrate over all space. Thus one obtains

$$\int d^3q \left[\psi_m^*\Delta\psi_n + \frac{2\mu}{\hbar^2}\,(W_n - U)\,\psi_m^*\psi_n\right] = 0\,.$$

According to Green's Theorem and with the orthogonality relation (2) we have

$$\delta_{mn}W_n = \int d^3q \frac{\hbar^2}{2\mu}\left[(\vec{\nabla}\psi_n).(\vec{\nabla}\psi_m^*) + U\psi_n\psi_m^*\right]\,. \tag{6}$$

[5]For simplicity I set the Dichtigkeitsfunktion equal to 1. (Sorry, I cannot translate this - can anyone help?)

Every energy level can be understood as a volume integral of the energy density. One can construct now the appropriate integral for any arbitrary function:

$$W = \int d^3q \left[\frac{\hbar^2}{2\mu} |\vec{\nabla}\psi|^2 + U|\psi|^2 \right] . \tag{7}$$

So one obtains through the use of (3) the expression

$$W = \sum_n |c_n|^2 W_n \, . \tag{8}$$

According to our interpretation of $|c_n|^2$, the right-hand side of (8) is the average value of the total energy of the atomic system. This average value can be represented as a volume integral of energy density of the function ψ. As long as we dwell on periodic processes, there is nothing else substantial to be said in favour of our Ansatz.

3 Aperiodic Systems

We therefore consider aperiodic processes and regard first for the sake of simplicity the case of linear motion with constant velocity along the x-axis. Here the differential equation reads:

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0\,; \qquad k^2 = \frac{2\mu}{\hbar^2} W \, . \tag{9}$$

Its eigenvalues W are all positive and it has eigenfuctions:

$$\psi = ce^{\pm ikx} \, .$$

In order to define weights and frequencies of occurrence, one must normalise all the eigenfunctions. The integral form of equation (2) fails. (The integral diverges.) It is therefore obvious to use the average value instead:

$$\lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{+a} |\psi(k,x)|^2 \, dx \;=\; \lim_{a\to\infty} \frac{c^2}{2a} \int_{-a}^{+a} e^{-ikx} e^{ikx} dx \;=\; 1 \, . \tag{10}$$

It follows that $c = 1$ and one has as normalised eigenfunctions:

$$\psi(k,x) = e^{\pm ikx} \, . \tag{11}$$

Any function of x can be constructed out of these. It is still necessary to select a calibration for the k-scale. It is necessary to determine on which portion [of the x-axis?] the weight 1 will fall. For this, one may view free motion as the limiting case of periodic motion; namely the normal modes of a finite piece of the x-axis. It is generally known that the number of normal modes per length Δx and per interval Δk is given by $\dfrac{\Delta k}{2\pi} = \Delta\left(\dfrac{1}{\lambda}\right)$ where λ is the wavelength. As one expects,

$$\psi(x) \;=\; \int_{-\infty}^{\infty} c(k)\psi(k,x) d\left(\frac{k}{2\pi}\right) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) e^{ikx} dk \tag{12}$$

with

$$c(-k) = c^*(k)\,. \tag{13}$$

Then $|c(k)|^2$ can be viewed as the measure of frequency of occurrence for the interval $\frac{dk}{2\pi}$. For a collection of atoms whose distribution is given by $c(k)$, the number of atoms is represented as in equation (4), by the integral

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx \;=\; \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dx \left| \int_{-\infty}^{\infty} c(k) e^{ikx} dk \right|^2 . \tag{14}$$

Consider the case where only a small interval $k_1 \le k \le k_2$ is occupied. Then

$$\int_{-\infty}^{\infty} c(k) e^{ikx} dk \;=\; \bar{c} \int_{k_1}^{k_2} e^{ikx} dk \;=\; \frac{\bar{c}}{ix} \left(e^{ik_2 x} - e^{ik_1 x} \right) ,$$

where $\bar{c}$ represents the average value. Therefore one has:

$$\begin{aligned} \int_{-\infty}^{\infty} |\psi(x)|^2\, dx &= \frac{|\bar{c}|^2}{4\pi^2} \int_{-\infty}^{\infty} \frac{dx}{x^2} \left(e^{ik_2 x} - e^{ik_1 x} \right) \left(e^{-ik_2 x} - e^{-ik_1 x} \right) \\ &= \frac{|\bar{c}|^2}{4\pi^2} \int_{-\infty}^{\infty} \frac{dx}{x^2} \sin\left(\frac{k_2 - k_1}{2} x \right) \;=\; \frac{|\bar{c}|^2}{2\pi} (k_2 - k_1)\,. \end{aligned}$$

Now the momentum of the translatory motion, belonging to the eigenfunctions (11), is given by de Broglie's equation

$$p \;=\; \frac{h}{\lambda} \;=\; \hbar k\,. \tag{15}$$

It is perhaps not superfluous to remark that p can be understood as a "matrix". One must therefore define the matrix in the regime of a continuous spectrum through average values rather than through integrals.

$$\begin{aligned} p(k,k') &= \frac{\hbar}{i} \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{+a} \psi^*(k,x) \frac{\partial \psi(k',x)}{\partial x} dx \\ &= \frac{\hbar}{i} \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{+a} e^{-ikx} ik' e^{-ik'x} dx\,. \end{aligned}$$

$$\Rightarrow p(k,k') = \begin{cases} \hbar k & \text{for} \quad k = k', \\ 0 & \text{for} \quad k \neq k'\,. \end{cases} \tag{16}$$

Finally, through $\frac{\Delta p}{\hbar}$, one sets $\Delta k = k_2 - k_1$, and so,

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = |\bar{c}|^2 \frac{\Delta p}{h}\,. \tag{17}$$

With this, one has the result that a box of dimensions $\Delta x = 1$ and $\Delta p = h$ has a weight of 1, which is in agreement with the multiply verified ansatz of Sackur and Tetrode[6]. The

[6] A. Sacker, Ann. d. Phys. **36**, 958, 1911; **40**, 67, 1913; H. Tetrode, Phys. ZS. **14**, 212, 1913; Ann. d. Phys. **38**, 434, 1912.

second result is that $|c(k)|^2$ is the frequency of occurrence of a motion whose momentum is $p = \hbar k$.

Now we turn our attention to accelerated motion. Here we can naturally define a certain distribution of events in an analogous manner. This is however not a rational [approach?] with regards to collision processes. With these processes, one has asymptotic, straight-line motion before and after a collision. The particles exist therefore in practically free states for long periods of time (as compared with the duration of the impact itself) before and after the collision. One comes therefore to the following understanding (which is in accordance with problems raised by experiments): If, for asymptotic motion before impact, the distribution function $|c(k)|^2$ is known, can one obtain from it the distribution function after impact?

Naturally, we are referring here to a stationary beam of particles. Mathematically, our task amounts to the following: The stationary oscillation field ψ must be divided into incoming and outgoing waves. These are asymptotic plane waves. One can represent both of these through Fourier integrals of the form (12) by choosing the coefficient function $c(k)$ for the incoming wave arbitrarily. Then it shall be shown that $c(k)$ for the outgoing wave is completely determined. $c(k)$ yields a distribution into which the given collection of particles is transformed by the impacts. To gain a clearer picture, we must first consider the 1-dimensional case.

4 Asymptotic Behaviour of Eigenfunctions of the Continuous Spectrum for One Degree of Freedom

The Schrödinger differential equation reads:

$$\frac{d^2\psi}{dx^2} + \frac{2\mu}{\hbar^2}\left(W - U(x)\right)\psi = 0\,, \tag{18}$$

where $U(x)$ represents the potential energy. For expediency, we set

$$\frac{2\mu}{\hbar^2}W = k^2 \quad \text{and} \quad \frac{2\mu}{\hbar^2}U(x) = V(x)\,. \tag{19}$$

Then we have

$$\frac{d^2\psi}{dx^2} + k^2\psi = V\psi\,. \tag{20}$$

We examine the asymptotic behaviour of the solution at infinity. To have simple relations, we assume that $V(x)$ falls off faster than $\dfrac{1}{x^2}$ as $x \to \infty$, ie:

$$|V(x)| < \frac{K}{x^2}\,, \tag{21}$$

where K is a positive number[7]. We now determine $\psi(x)$ through an itterative process. We start with

$$u_0(x) = e^{ikx}\,. \tag{22}$$

[7]With this assumption, the cases of pure Coulomb and dipole fields are excluded.

Then $u_1(x),\ u_2(x), \ldots$ are solutions of the approximation equation

$$\frac{d^2u_n}{dx^2} + k^2 u_n = V u_{n-1}\,,$$

which vanish as $x \to \infty$.

Then

$$u_n(x) = \frac{1}{k}\int_x^\infty u_{n-1}(\xi) V(\xi) \sin k(\xi - x) d\xi\,,$$

as one can directly verify. One has

$$u_n(x) \leq \frac{1}{k}\int_x^\infty |u_{n-1}(\xi)|\,.\,|V(\xi)|\,d\xi\,.$$

We show now that

$$|u_{n-1}(\xi)| \leq \frac{1}{n!}\left(\frac{K}{kx}\right)^n.$$

For $n = 0$, this equation is true since, in agreement with (22), $|u_0(x)| \leq 1$. We assume now that it is correct for $n-1$:

$$|u_{n-1}(\xi)| \leq \frac{1}{(n-1)!}\left(\frac{K}{k\xi}\right)^{n-1};$$

it then follows

$$|u_{n-1}(x)| \leq \frac{1}{k}\frac{1}{(n-1)!}\left(\frac{K}{k}\right)^{n-1} K\int_x^\infty \xi^{-n+1}\xi^{-2} d\xi = \frac{1}{n!}\left(\frac{K}{kx}\right)^n,$$

as has been stated. Therefore the series

$$\psi(x) = \sum_{n=0}^{\infty} u_n(x) \tag{23}$$

converges uniformly for any finite interval of x. $\psi(x)$ lends itself to being differentiated term by term arbitrarily often, and is thus, as can clearly be seen, the desired solution of our differential equation.

However, since all terms $u_1, \ldots, u_n$ vanish as $x \to \infty$, the function ψ asymptotically approaches $u_0(x) = e^{ikx}$ as $x \to \infty$. In a similar way, one sees that another solution exists. This solution asymptotically approaches e^{-ikr} as $x \to \infty$. Since the general solution has only two constants, the solution for $x \to \infty$ must have the form

$$\psi^+ = ae^{ikx} + be^{-ikx}. \tag{24}$$

Here the degeneracy of the system becomes obvious. To each energy W, there are associacted two values of k ($\pm k$) and two linearly independent solutions.

It follows similarly that the general solution for $x \to -\infty$ must have the same form:

$$\psi^- = Ae^{ikx} + Be^{-ikx}. \tag{25}$$

Therefore the amplitutes A and B are definite functions of a and b.

We separate the solution into incoming and outgoing waves and we include a time factor e^{ikvt} ($kv = 2\pi\nu = \frac{W}{\hbar}$) and set

$$\left.\begin{aligned} a = c_i e^{i\varphi_i t}, \quad A = C_o e^{i\phi_o t}, \\ b = c_o e^{-i\varphi_o t}, \quad B = C_i e^{-i\phi_i t}. \end{aligned}\right\} \tag{26}$$

Then

$$\left.\begin{aligned} \psi^+(x) = c_i e^{ik(x+vt+\varphi_i)} + c_o e^{-ik(x-vt+\varphi_o)}, \\ \psi^-(x) = C_o e^{ik(x+vt+\phi_o)} + C_i e^{-ik(x-vt+\phi_i)}. \end{aligned}\right\} \tag{27}$$

The respective parts are distinguished with indices i and o. i denotes the incoming wave and o denotes the outgoing wave. We are interested in the case where only one wave arrives at $x = +\infty$. In this case $C_i = 0$. Moreover, one can arbitrarily set $\varphi_i = 0$. Then one obtains:

$$\left.\begin{aligned} \psi^+(x) = c_i e^{ik(x+vt)} + c_o e^{-ik(x-vt+\varphi_o)}, \\ \psi^-(x) = C_o e^{ik(x+vt+\phi_o)}. \end{aligned}\right\} \tag{28}$$

It has been seen that through the integration, ψ^- is determined by ψ^+, ie: A and B are particular functions of a and b. In the case $C_i = 0$, $B = 0$ also; in addition we have two equations of the form

$$\left.\begin{aligned} A = A(a, b), \\ 0 = B(a, b). \end{aligned}\right\} \tag{29}$$

From these two, one can express b in terms of a and thus express A in terms of a alone. This means however that the constants of the reflected and transmitted waves can be calculated from the amplitude of the incident wave. One can now see that a relation exists between the intensities of the three waves. The simplest way to obtain this relationship is through energy conservation.

5 The Principle of Conservation of Energy

To derive this principle, we must return to the form of the Schrödinger equation which does not presuppose purely periodic oscillations. The wave equation takes on the form:

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{30}$$

Here v is the wave velocity. One arrives to the Schrödinger equation with the help of de Broglie[8]:

$$\begin{aligned} h\nu = W = \frac{\mu}{2}u^2 + U, \\ v = \lambda\nu, \\ \frac{h}{\lambda} = p = \mu u. \end{aligned}$$

[8]We disregard Special Relativity and calculate with Classical Mechanics

Then we have

$$\frac{1}{v^2} = \frac{h^2}{\lambda^2}\frac{1}{h^2\nu^2} = \frac{\mu^2 u^2}{W^2} = \frac{\frac{\mu}{2}u^2.2\mu}{W^2} = \frac{2\mu}{W^2}(W-U). \tag{31}$$

One now seeks a solution whose time dependence is give by the factor $e^{2\pi i\nu t} = e^{iWt/\hbar}$. So one obtains:

$$\frac{d^2\psi}{dx^2} + \frac{2\mu}{\hbar^2}(W-U) = 0.$$

We consider however the general form (30) and multiply the equation by $\dfrac{\partial\psi}{\partial t}$:

$$\frac{\partial^2\psi}{\partial x^2}\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}\right) - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}\right) - \frac{\partial}{\partial t}\frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2.$$

When v depends only on x, we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}\right) - \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 + \frac{1}{2v^2}\left(\frac{\partial\psi}{\partial t}\right)^2\right] = 0. \tag{32}$$

Integrating over all space, one obtains:

$$\left[\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}\right]_{-\infty}^{\infty} - \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{1}{2}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2 + \frac{1}{v^2}\left(\frac{\partial\psi}{\partial t}\right)^2\right\}dx = 0 \tag{33}$$

As was show in section 2, the volume integral has to be interpretted as the total energy available in space. However, the expression does not interest us because we are concerned with the in and out-flow of energy. This is represented by the surface term in equation (33). The time average of the second term vanishes for a periodic process. One obtains by use of the notations introduced in equations (24) and (25)

$$\overline{\frac{\partial\psi^-}{\partial x}\frac{\partial\psi^-}{\partial t}} = \overline{\frac{\partial\psi^+}{\partial x}\frac{\partial\psi^+}{\partial t}}. \tag{34}$$

This equation demonstrates that the incoming and outgoing energies are equal. By inserting here the real parts of (27) we obtain

$$C_o^2 - C_i^2 = c_i^2 - c_o^2, \tag{35}$$

or in the case $C_i = 0$ (as in equation (28)):

$$c_i^2 = c_o^2 + C_o^2. \tag{36}$$

This means however that for any wave front of a given k, the incoming intensity is split up into both left and right scattered waves. Or in the language of corpuscular theory: If a particle of given energy strikes an atom, it is either reflected or transmitted. The sum of probabilities for both these events is 1.

The principle of conservation of energy implies therefore the conservation of number of particles. The reason for this lies in the degeneracy of the system. There is more than one path associated with the same energy and these paths are set in relation with each other.

6 Generalisation to Three Degrees of Freedom. Motion With Constant Velocity

We consider now the motion of particles in space under the influence of a potential energy $U(x, y.z)$. Similarly to (30), one has the differential equation

$$\Delta\psi - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0, \tag{37}$$

where v (in approximation with Classical Mechanics) is given again by equation (31). Here the conservation principle reads:

$$\vec{\boldsymbol{\nabla}}.\left(\frac{\partial\psi}{\partial t}\vec{\boldsymbol{\nabla}}\psi\right) - \frac{\partial}{\partial t}\frac{1}{2}\left[(\vec{\boldsymbol{\nabla}}\psi)^2 + \frac{1}{v^2}\left(\frac{\partial^2\psi}{\partial t^2}\right)\right] = 0, \tag{38}$$

or integrating over all space,

$$\int_\infty \frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial n}ds - \frac{\partial}{\partial t}\int\frac{1}{2}\left[(\vec{\boldsymbol{\nabla}}\psi)^2 + \frac{1}{v^2}\left(\frac{\partial\psi}{\partial t}\right)^2\right]d^3r = 0, \tag{39}$$

where ds is an infinite closed surface with outer normal $\hat{\boldsymbol{n}}$. For periodic processes it follows that the time average

$$\overline{\int_\infty \frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial n}ds} = 0. \tag{40}$$

In this case the differential equation reads

$$\Delta\psi + (k^2 - V)\psi = o, \tag{41}$$

where

$$k^2 = \frac{2\mu}{\hbar^2}W, \qquad V(\vec{\boldsymbol{r}}) = \frac{2\mu}{\hbar^2}U(\vec{\boldsymbol{r}}). \tag{42}$$

The differential equation for unaccelerated motion is given by

$$\Delta\psi + k^2\psi = 0, \tag{43}$$

with the solution:

$$\psi = e^{i\vec{\boldsymbol{k}}.\vec{\boldsymbol{r}}}. \tag{44}$$

Here $\vec{\boldsymbol{r}}$ is the vector (x, y, z) and the vector $\vec{\boldsymbol{k}}$ satisfies the equation

$$\left|\vec{\boldsymbol{k}}\right|^2 = k_x^2 + k_y^2 + k_z^2 \equiv k^2. \tag{45}$$

It is equal, up to a factor, to the momentum vector:

$$\vec{\boldsymbol{p}} = \hbar\vec{\boldsymbol{k}}. \tag{46}$$

The de Broglie wavelength is given by $\frac{h}{\lambda} = p = |\vec{\boldsymbol{p}}| = \hbar k$. The solution (44) should be seen as normalised in the sense of an average (see (10)).

The general solution of (43) is

$$\psi(\vec{\boldsymbol{r}}) = u_0(\vec{\boldsymbol{r}}) = \int c(\hat{\boldsymbol{s}}) e^{ik(\vec{\boldsymbol{r}}.\hat{\boldsymbol{s}})} d\omega; \qquad c(\hat{\boldsymbol{s}}) = c^*(\hat{\boldsymbol{s}}), \tag{47}$$

where $\hat{\boldsymbol{s}}$ is a unit vector and $d\omega$ is an element of solid angle. This represents unaccelerated [inertial] motion in all possible directions with the same energy. According to our principle, $|c(\hat{\boldsymbol{s}})|^2$ computes the number of particles flying in direction $\hat{\boldsymbol{s}}$ per unit solid angle.

We want to deduce an asymptotic representation for u_0 which will clearly show the behaviour of u_0 at infinity. Although it is very simple to obtain this result, we want to obtain it here by means of a more general method; one which can be transferred later to handle more complicated cases. We consider here a new orthogonal coördinate system introduced with the help of the orthogonal transformation:

$$\left.\begin{aligned} x &= a_{11}X + a_{12}Y + a_{13}Z, \quad X = a_{11}x + a_{21}y + a_{31}z, \\ y &= a_{21}X + a_{22}Y + a_{23}Z, \quad Y = a_{12}x + a_{22}y + a_{32}z, \\ z &= a_{31}X + a_{32}Y + a_{33}Z, \quad Z = a_{13}x + a_{23}y + a_{33}z \end{aligned}\right\} \tag{48}$$

At the same time we introduce a new unit vector $\hat{\boldsymbol{S}}$ in place of $\hat{\boldsymbol{s}}$ with the aid of the same orthogonal transformation. Then the volume angle element $d\omega$ becomes $d\Omega$ and

$$\vec{\boldsymbol{r}}.\hat{\boldsymbol{s}} = \vec{\boldsymbol{R}}.\hat{\boldsymbol{S}}. \tag{49}$$

Now we choose the new coördinate system in particular so that

$$X = 0, \quad Y = 0, \tag{50}$$

and so

$$Z = r = \sqrt{x^2 + y^2 + z^2}. \tag{51}$$

Our integral becomes

$$u_0(x, y, z) = u_0(a_{13}Z, a_{23}Z, a_{33}Z) = \int d\Omega c(a_{11}S_x + a_{12}S_y + a_{13}S_z, \ldots) e^{ikZS_z}.$$

Now we introduce polar coördinates for $\hat{\boldsymbol{S}}$:

$$S_x = \sin\vartheta\cos\varphi, \quad S_y = \sin\vartheta\sin\varphi, \quad S_z = \cos\vartheta, \tag{52}$$

and setting $\cos\vartheta = \mu$;

$$u_0 = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\mu c \left(\sqrt{1-\mu^2}(a_{11}\cos\varphi + a_{12}\sin\varphi) + \mu a_{13}, \ldots\right) e^{ikZ\mu}.$$

Through partial integration

$$u_0 = \frac{1}{ikZ}\int_0^{2\pi} d\varphi \left[c(a_{13}, a_{23}, a_{33})e^{ikZ} - c(-a_{13}, -a_{23}, -a_{33})e^{-ikZ}\right]$$

$$-\frac{1}{ikZ}\int_0^{2\pi} d\varphi \frac{d}{d\mu} c\left(\sqrt{1-\mu^2}(a_{11}\cos\varphi + a_{12}\sin\varphi) + \mu a_{13}, \ldots\right) e^{ikZ\mu} d\mu.$$

Repeated application of the same process shows that the second term vanishes as Z^{-2}. One inserts now $Z = r,\ a_{13} = \dfrac{x}{Z} = \dfrac{x}{r}, \ldots$, and so one obtains the asymptotic representation

$$u_0^{\infty}(x, y, z) = \frac{2\pi}{ikr}\left[c\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) e^{ikr} - c\left(-\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right) e^{-ikr}\right], \tag{53}$$

or in Euler notation with $c = |c|e^{ik\gamma}$:

$$u_0^{\infty}(\vec{\boldsymbol{r}}) = \frac{4\pi}{k}\left|c\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)\right| \frac{\sin\left[k\left(r + \gamma\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)\right)\right]}{r}. \tag{54}$$

This means that u_0 behaves asymptotically like a spherical wave whose amplitude and phase depend on direction. The intensity as a function of $\hat{\boldsymbol{s}}$ determines the rate of incoming particles passing through a solid angle element $d\omega$ in the direction $\hat{\boldsymbol{s}}$.

7 Elastic Collisions

We turn our attention now to the integration of the general equation (41).

$$\Delta\psi + (k^2 - V)\psi = 0. \tag{55}$$

This represents the physical case where a single electron collides with an unexcited atom. As in section 4, we determined ψ through an itterative procedure. This served as a starting point for the function u_0 (equation (47)). So then we can calculate $u_1, u_2, \ldots$ one after another from the approximation equation

$$\Delta u_n n + k^2 u_n = V u_{n-1} = F_{n-1}. \tag{56}$$

Green's Theorem yields the solution which is equivalent to an outgoing wave with time factor e^{ikvt}:

$$u_n(\vec{\boldsymbol{r}}) = -\frac{1}{4\pi}\int F_{n-1}(\vec{\boldsymbol{r}}')\frac{e^{-ik|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} d^3r'. \tag{57}$$

The convergence of the procedure can be proven based on the assumption that V falls off[9] as $\dfrac{1}{r^2}$. We do not focus on this however. Instead we assume that a solution may be represented by a series

$$\psi(\vec{\boldsymbol{r}}) = \sum_{n=0}^{\infty} u_n(\vec{\boldsymbol{r}}).$$

[9]Herewith, the case of ions is excluded. Concerning this, one must take as a starting point for our approximation procedure, a hyperbolic path rather than linear motion. See here also the the soon to be appearing publication of J.R. Oppenheimer, Proc. Cambridge Phil. Soc. 26 July 1926.

We examine the asymptotic behaviour of $u_n(\vec{\boldsymbol{r}})$. We write explicitly:

$$u_n(\vec{\boldsymbol{r}}) = -\frac{1}{4\pi}\int F_{n-1}(x',y',z')\frac{e^{-ik\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}dx'dy'dz'.$$

As in section 6, we rotate the coördinate system and apply the same rotation to the integration variables. We have then

$$u_n(x,y,z) = u_n(a_{13}Z, a_{23}Z, a_{33}Z) = -\frac{1}{4\pi}\int F'_{n-1}(X',Y',Z')\frac{e^{-ik\sqrt{X'^2+Y'^2+Z'^2}}}{\sqrt{X'^2+Y'^2+(Z-Z')^2}}dX'dY'dZ'. \tag{58}$$

Therefore

$$F'_{n-1}(X',Y',Z') = F_{n-1}(a_{11}X' + a_{12}Y' + a_{13}Z', \ldots, \ldots). \tag{59}$$

Now introduce polar coördinates

$$X' = \rho\sin\vartheta\cos\varphi; \quad Y' = \rho\sin\vartheta\cos\varphi; \quad Z' = \rho\cos\vartheta.$$

Then

$$u_n = -\frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\infty}\rho^2 d\rho \int_0^{\pi}\sin\vartheta d\vartheta F'_{n-1}(\rho\sin\vartheta\cos\varphi,\ldots)\frac{e^{-ik\sqrt{\rho^2+Z^2-2\rho Z\cos\vartheta}}}{\sqrt{\rho^2+Z^2-2\rho Z\cos\vartheta}}.$$

Instead of ϑ, we introduce the integration variable μ through

$$\sqrt{\rho^2+Z^2-2\rho Z\cos\vartheta} = Z\mu, \quad \sin\vartheta d\vartheta = \frac{Z}{\rho}\mu d\mu.$$

The limits of integration become

$$\vartheta = 0 \ : \ \mu = \left|\frac{\rho}{Z} - 1\right|; \qquad \vartheta = \pi \ : \ \mu = \frac{\rho}{Z} + 1$$

and $\cos\vartheta$ and $\sin\vartheta$ become special functions $c(\rho,Z,\mu)$ and $s(\rho,Z,\mu)$. $c(\rho,Z,\mu)$ and $s(\rho,Z,\mu)$ take on values $c = 1$, $s = 0$ in the lower limit and $c = -1$, $s = 0$ in the upper limit. Thus one obtains

$$u_n = -\frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\infty}\rho d\rho \int_{\left|\frac{\rho}{Z}-1\right|}^{\frac{\rho}{Z}+1} F'_{n-1}(\rho s\cos\varphi, \rho s\sin\varphi, \rho c)e^{-ik\mu Z}d\mu.$$

Through partial integration we obtain the asymptotic representation as in section 6:

$$u_n^{\infty} = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\infty}\rho d\rho \frac{1}{ikZ}\left[F'_{n-1}(0,0,\rho)e^{-ik(Z+\rho)} - F'_{n-1}(0,0,-\rho)e^{-ik|Z-\rho|}\right].$$

Using equation (59)

$$F'_{n-1}(0,0,\rho) = F_{n-1}(a_{13}\rho, a_{23}\rho, a_{33}\rho) = F_{n-1}\left(\frac{\rho x}{r}, \frac{\rho y}{r}, \frac{\rho z}{r}\right),$$

$$F'_{n-1}(0,0,-\rho) = F_{n-1}(-a_{13}\rho, -a_{23}\rho, -a_{33}\rho) = F_{n-1}\left(-\frac{\rho x}{r}, -\frac{\rho y}{r}, -\frac{\rho z}{r}\right).$$

So we obtain:

$$u_n^\infty = \frac{e^{-ikr}}{2ikr}\int_0^\infty \rho d\rho F_{n-1}\Big(\frac{\rho x}{r}, ..\Big)e^{-ik\rho} - \frac{e^{-ikr}}{2ikr}\int_0^r \rho d\rho F_{n-1}\Big(-\frac{\rho x}{r}, ..\Big)e^{ik\rho} - \frac{e^{ikr}}{2ikr}\int_r^\infty \rho d\rho F_{n-1}\Big(-\frac{\rho x}{r}, ..\Big)e^{-ik\rho}.$$

Here the last integral vanishes as $r \to \infty$. In explaining why this term vanishes, we use as a starting point $|V| \leq ar^{-2}$ and so, because $|u_0| \leq br^{-1}$, we have $|F_{n-1}| \leq \frac{A}{r^3}$, and thus

$$\left|\int_r^\infty \rho d\rho F_{n-1}\left(-\frac{\rho x}{r}, \ldots\right)e^{-ik\rho}\right| \leq A\int_r^\infty \frac{d\rho}{\rho^2} = \frac{A}{r}.$$

With this, we finally obtain

$$u_n^\infty = \frac{e^{-ikr}}{2ikr}\int_0^\infty \rho d\rho \left[F_{n-1}\left(\frac{\rho x}{r}, \ldots\right)e^{-ik\rho} - F_{n-1}\left(-\frac{\rho x}{r}, \ldots\right)e^{-ik\rho}\right]. \tag{60}$$

This expression can be brought into a more transparent form. To that end, we introduce the Fourier co-efficients of the function F_{n-1}:

$$f_{n-1}(\vec{\boldsymbol{k}}) = \frac{1}{(2\pi)^3}\int F_{n-1}(\vec{\boldsymbol{r}})e^{-i\vec{\boldsymbol{k}}.\vec{\boldsymbol{r}}}d^3r = \frac{1}{(2\pi)^3}\int_0^\infty r^2dr\int d\omega F_{n-1}(r\hat{\boldsymbol{s}})e^{-ir(\vec{\boldsymbol{k}}.\hat{\boldsymbol{s}})}. \tag{61}$$

By the procedure already used, we determine the asymptotic value and obtain:

$$f_{n-1}^\infty(\vec{\boldsymbol{k}}) = \frac{1}{4\pi^2 ik}\int_0^\infty rdr\left[F_{n-1}\left(\frac{rk_x}{k}, \ldots\right)e^{ikr} - F_{n-1}\left(-\frac{rk_x}{k}, \ldots\right)e^{-ikr}\right].$$

Therefore

$$f_{n-1}^\infty(-k\hat{\boldsymbol{s}}) = \frac{1}{4\pi^2 ik}\int_0^\infty \rho d\rho \left[F_{n-1}\left(\frac{\rho x}{r}, \ldots\right)e^{-ik\rho} - F_{n-1}\left(-\frac{\rho x}{r}, \ldots\right)e^{ik\rho}\right]. \tag{62}$$

Inserting equation (60), we finally obtain

$$u_n^\infty(\vec{\boldsymbol{r}}) = 2\pi^2 f_{n-1}^\infty(-k\hat{\boldsymbol{s}})\frac{e^{-ikr}}{r}. \tag{63}$$

Comparing this with equations (47) and (54) we can see tha an observer standing at infinity will recognise the scattered radiation as a plane wave whose amplitude

$$\frac{k}{2\pi}2\pi^2\left|f_{n-1}^\infty(-k\hat{\boldsymbol{s}})\right| = k\pi\left|f_{n-1}^\infty(-k\hat{\boldsymbol{s}})\right|$$

depends on the direction $\hat{\boldsymbol{s}}$. Therefore the probablility that an electron with direction $\hat{\boldsymbol{s}}$ is scattered into a solid angle element $d\omega$, is given by

$$\Phi d\omega = \pi^2 k^2\left|\sum_{n=0}^\infty f_n^\infty(-k\hat{\boldsymbol{s}})\right|^2 d\omega. \tag{64}$$

The complete solution has the asymptotic form:

$$\psi^{\infty} = u_0^{\infty} + \sum_{n=1}^{\infty} u_n^{\infty} = \frac{2\pi}{k}\left[|c(\hat{\boldsymbol{s}})|e^{ik(r+\delta)} + k\pi \sum_{n=1}^{\infty} f_n^{\infty}(-k\hat{\boldsymbol{s}})e^{-ikr}\right].$$

One includes here a time factor e^{ikvt} and so one easily obtains "conservation of number of particles" from equation (40).

To first approximation one has

$$\Phi d\omega = \pi^2 k^2 \left|f_0^{\infty}(-k\hat{\boldsymbol{s}})\right|^2 d\omega, \tag{65}$$

where one either strictly calculates f_0 from the formula

$$f_0(\vec{\boldsymbol{k}}) = \frac{1}{(2\pi)^2}\int F_0(\vec{\boldsymbol{r}})e^{-i\vec{\boldsymbol{k}}.\vec{\boldsymbol{r}}}d^3r, \tag{66}$$

or one immediately makes use of the expression given by (62):

$$f_0^{\infty}(-k\hat{\boldsymbol{s}}) = \frac{1}{4\pi^2 ik}\int_0^{\infty} \rho d\rho \left[F_0\left(\rho\hat{\boldsymbol{s}}\right)e^{ik\rho} - F_0\left(-\rho\hat{\boldsymbol{s}}\right)e^{-ik\rho}\right]. \tag{67}$$

8 Inelastic Electron Collisions

An atom (or a molecule, but we prefer to always speak of an "atom") is represented by the Hamiltonian function $H^a(p,q)$[10]. If the Schrödinger differential equation has been solved, the eigenvalues W_n^a and eigenfunctions $\psi_n^a(q)$ identically satisfy the equations

$$[H^a - W_n^a, \psi_n^a] = 0. \tag{68}$$

An electron collides with an atom. The Hamiltonian function of the free electron is

$$H^{\varepsilon} = \frac{1}{2\mu}\left(p_x^2 + p_y^2 + p_z^2\right).$$

Its eigenvalues W^{ε} are all positive and the eigenfuctions are

$$e^{\pm kr}\ (^{11}) \quad \text{where} \quad k^2 = \frac{2\mu}{\hbar^2}W^{\varepsilon}. \tag{69}$$

The general solution which corresponds to the incoming wave is

$$\psi_k^{\varepsilon} = \int_{\vec{\boldsymbol{r}}.\hat{\boldsymbol{s}}>0} c^0(\hat{\boldsymbol{s}})e^{ik(\vec{\boldsymbol{r}}.\hat{\boldsymbol{s}})}d\omega. \tag{70}$$

It satisfies the differential equation

$$[H^{\varepsilon} - W^{\varepsilon}, \psi_k^{\varepsilon}] = 0 \quad \text{or} \quad \Delta\psi_k^{\varepsilon} + k^2\psi_k^{\varepsilon} = 0. \tag{71}$$

[10] We write in abbreviated form p, q in place of $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$.

[11] This must be a misprint in the original paper. It should read $e^{\pm ikr}$.

The potential energy of the atom-electron interaction is

$$U(q; x, y, z). \tag{72}$$

The interaction between the two particles leads to the Hamiltonian function

$$H = H^0 + \lambda H^{(1)},$$

where

$$H^0 = H^a + H^\varepsilon \quad \text{and} \quad \lambda H^{(1)} = U.$$

The undisturbed system has the solution

$$W^0_{nk} = W^a_n + W^\varepsilon, \qquad \psi^0_{nk} = \psi^a_n \psi^\varepsilon_k.$$

We solve the Schrödinger differential equation of the perturbed system

$$[H - W, \psi]$$

through the Ansatz

$$\psi = \psi^0 + \lambda \psi^{(1)} + \dots$$

The one obtains the approximation equations

$$\begin{array}{rcl} \left[H^0 - W^0_{nk}, \psi^{(1)}_{nk}\right] & = & -U\psi^0_{nk}, \\ \left[H^0 - W^0_{nk}, \psi^{(2)}_{nk}\right] & = & -U\psi^{(1)}_{nk}, \\ \text{.....} & & \text{.....} \end{array}$$

whose left hand sides are in agreement with each other. We write explicitly:

$$\left[H^a, \psi^{(1)}_{nk}\right] + \left[H^\varepsilon, \psi^{(1)}_{nk}\right] - W^0_{nk}\psi^{(1)}_{nk} = -U\psi^0_{nk},$$

or

$$\left[H^a, \psi^{(1)}_{nk}\right] - \frac{\hbar^2}{2\mu}\Delta\psi^{(1)}_{nk} - W^{(0)}\psi^{(1)}_{nk} = -U\psi^0_{nk}.$$

We seek to solve this equation through the Ansatz:

$$\psi^{(1)}_{nk} = \sum_m u^{(1)}_{nm}(\vec{\boldsymbol{r}})\psi^a_m,$$

ie: through the expansion in eigenfuctions of the unperturbed atom whose coefficients are still undetermined functions of the position vector $\vec{\boldsymbol{r}}$ of the electron.

Now, according to (68),

$$\begin{array}{rcl} \left[H^a, \psi^{(1)}_{nk}\right] & = & \sum_m u^{(1)}_{nm}(\vec{\boldsymbol{r}}) \left[H^a, \psi^a_m\right] \\ & = & \sum_m u^{(1)}_{nm}(\vec{\boldsymbol{r}}) W^a_m \psi^a_m. \end{array}$$

The right hand side of the equation can be expanded in the same manner:

$$U\psi^0_{nk} = \psi^\varepsilon_k U\psi^a_n = \psi^\varepsilon_k \sum_m U_{nm}\psi^a_m.$$

The co-efficients U_{nm} make up a matrix which represents the potential energy. We insert this expression into the differential equation and obtain:

$$\sum_m \psi^a_m \left[u^{(1)}_{nm}(\vec{\boldsymbol{r}})W^a_m - \frac{\hbar^2}{2\mu}\Delta u^{(1)}_{nm}(\vec{\boldsymbol{r}}) - u^{(1)}_{nm}(\vec{\boldsymbol{r}})\left(W^a_m + W^\varepsilon\right)\right] = -\sum_m \psi^a_m U_{nm}\psi^\varepsilon_k.$$

By equating co-efficients of ψ^a_m one obtains a differential equation for $u^{(1)}_{nm}(\vec{\boldsymbol{r}})$. We multiply this equation by $-\dfrac{2\mu}{\hbar^2}$, and using the abbreviations

$$V = \frac{2\mu}{\hbar^2}U, \qquad V_{nm} = \frac{2\mu}{\hbar^2}U_{nm}, \tag{73}$$

$$k^2_{nm} = \frac{2\mu}{\hbar^2}\left(W^a_n - W^a_m + W^\varepsilon\right) = \frac{2\mu}{\hbar^2}\left(h\nu^a_{nm} + W^\varepsilon\right), \tag{74}$$

we have

$$\Delta u^{(1)}_{nm} + k^2_{nm}u^{(1)}_{nm} = V_{nm}\psi^\varepsilon_k. \tag{75}$$

In this way we have related the problem to the earlier treatment of inelastic collisions, since all following approximations lead to the same wave equation. The difference, however between this case and the earlier one is as follows: Any transition $(n \to m)$ of an atom corresponds to a distinct [unique?] differential equation whose right hand side is determined by the corresponding matrix element of the potential energy. Furthermore, the k value of the incoming wave is always replaced by another k value, k_{nm}, whose energy corresponds to

$$W^\varepsilon_{nm} = \frac{\hbar^2}{2\mu}k^2_{nm} = h\nu^a_{nm} + W^\varepsilon. \tag{76}$$

Already the fundamental qualitative laws of electron scattering follow: The energy of an electron after a collision is not generally equal to its energy before, but the energy of the atom differs by an energy step of $h\nu^a_{nm}$. For any collision process, a corresponding probability function

$$\Phi_{nm} = \pi^2 k^2_{nm}\left|f^\infty_0(-k_{nm}\hat{\boldsymbol{s}})\right|^2 \tag{77}$$

may be computed with the aid of equations (66) and (67).

9 Physical Conclusions

First we see that our formulas correctly describe the qualitative behaviour or atoms in collisions, therefore also the fact of "energy limits [levels?]". These energy limits [levels?] are always seen as the cornerstone of Quantum Mechanics and the grossest violation of Classical Mechanics. We arrange the energy levels of the atom from smallest to largest:

$$W^a_0 < W^a_1 < W^a_2 < \dots$$

The index 0 labels the ground state and so

$$h\nu^a_{nm} = W^a_n - W^a_m > 0 \quad \text{for} \quad n > m.$$

We consider first the case where the atom starts in its ground state. Then $\nu^a_{m0} > 0$ for all m and it follows from equation (76) that

$$W^\varepsilon_{0m} = W^\varepsilon - h\nu^a_{m0}.$$

If $W^\varepsilon < h\nu^a_{10}$, W^ε_{0m} would become negative for $m > 0$, which is impossible. Thus $m = 0$ and

$$W^\varepsilon_{00} = W^\varepsilon.$$

Elastic reflection takes place with [yield?] Φ_{00}. Let W^ε increase until

$$h\nu^a_{10} < W^\varepsilon < h\nu^a_{20}.$$

Now W^ε_{0m} is only positive for $m = 0$ and $m = 1$. Therefore one either has elastic reflection with yield Φ_{00}, or resonance excitation with yield Φ_{01}. If we increase W^ε further until

$$h\nu^a_{20} < W^\varepsilon < h\nu^a_{30},$$

then we have three cases: Elastic reflection with yield Φ_{00}, excitation of the first quantum jump with yield Φ_{01}, or excitation of the second quantum jump with yield Φ_{02}. One can continue in the same way.

Now consider the case where the atom starts in its second quantum state ($n = 1$). Then $\nu^a_{10} > 0$ and $\nu^a_{1m} < 0$ for $m = 2, 3, \ldots$ One has therefore

$$\begin{aligned} W^\varepsilon_{10} &= W^\varepsilon + h\nu^a_{10}, \\ W^\varepsilon_{11} &= W^\varepsilon, \\ W^\varepsilon_{1m} &= W^\varepsilon - h\nu^a_{1m}, \quad m = 2, 3, \ldots \end{aligned}$$

If $W^\varepsilon < h\nu^a_{21}$ then W^ε_{1m} is negative for $m = 2, 3, \ldots$ Therefore either a collision of the second type with energy gain of the electron of $h\nu^a_{10}$ and yield Φ_{10} or elastic reflection with yield Φ_{11} exists. If

$$h\nu^a_{21} < W^\varepsilon < h\nu^a_{31},$$

then the state $n = 2$ contributes to these processes with yield Φ_{12}. And so it goes on ...

In the general case, if the atom starts in the nth state, for

$$W^\varepsilon < h\nu^a_{n+1,n}$$

there are only collisions of the second type. Here the atom can drop into states $0, 1, \ldots, n-1$ and transfers energies $h\nu^a_{n0}, h\nu^a_{n0}, \ldots, h\nu^a_{n,n-1}$ to the electron with yields $\Phi_{n0}, \Phi_{n1}, \ldots, \Phi_{n,n-1}$ and elastic reflection Φ_{nn}. Increase W^ε above $h\nu^a_{n+1,n}$ so that

$$h\nu^a_{n+1,n} < W^\varepsilon < h\nu^a_{m+1,n},$$

then excitations contribute with yields $\Phi_{n,n+1}, \Phi_{n,n+2}, \ldots, \Phi_{n,m}$.

The next task would be to discuss the formula for the yields (77). However we will have to be satisfied here with a very preliminary [provisional?] and probably quite controversial picture. We assume that the potential U can be expanded in a power series in $\frac{1}{r}$. For a neutral atom we have to first approximation the dipole equation:

$$U(x,y,z) = \frac{e}{r^3} B\vec{\boldsymbol{r}}, \tag{78}$$

where $B(q)$ is the electrical moment of the atom. We represent this with a matrix B_{nm}. Then according to (73),

$$V_{nm} = \frac{2\mu c}{\hbar^2}\left(B_{nm}\frac{\vec{\boldsymbol{r}}}{r^3}\right). \tag{79}$$

Naturally this ansatz can only be correct for electrons which pass by the atom at a considerable distance. Our view is therefore limited to such electrons where[12] $r > r_0$. Therefore from equation (67)

$$f_0^\infty(-k_{nm}\hat{\boldsymbol{s}}) = \frac{1}{4\pi^2 i k_{nm}}\int_0^\infty \rho d\rho \left[F_{nm}\left(\rho\hat{\boldsymbol{s}}\right)e^{-ik_{nm}\rho} - F_{nm}\left(-\rho\hat{\boldsymbol{s}}\right)e^{ik_{nm}\rho}\right].$$

We assume that the incoming electrons constitute a parallel bundle corresponding to a plane wave. Then

$$F_{nm}(\rho\hat{\boldsymbol{s}}) = V_{nm}e^{ik\rho\hat{\boldsymbol{z}}} = \frac{2\mu e}{\hbar^2}\left(B_{nm},\hat{\boldsymbol{s}}\right)\frac{e^{ik\rho\hat{\boldsymbol{z}}}}{\rho^2}.$$

Now

$$i\pi k_{nm} f_0^\infty(-k_{nm}\hat{\boldsymbol{s}}) = \frac{\mu e}{\pi\hbar^2}\left(B_{nm},\hat{\boldsymbol{s}}\right)A, \tag{80}$$

where, with $\hat{\boldsymbol{z}} = \cos\vartheta$;

$$A = \int_{r_0}^\infty \frac{d\rho}{\rho}\cos\left[\rho(k\cos\vartheta - k_{nm})\right], \tag{81}$$

or

$$A = C_i\left(r_0[k\cos\vartheta - k_{nm}]\right), \tag{82}$$

where $C_i(x)$ is the integral cosine[13].

Therefore from equation (77), the yield function becomes

$$\Phi_{nm} = \left(\frac{\mu e}{\pi\hbar^2}\right)^2 \left|B_{nm},\hat{\boldsymbol{s}}\right|^2 A^2. \tag{83}$$

Finally, one averages over all positions of the atom and so the products of pairs of components of B_{nm} all vanish. The averages of squares of components of B_{nm} become equal to $\frac{1}{3}|P_{nm}|^2$ where P denotes the magnitudes of the electrical moments. So one obtains:

$$\Phi_{nm} = \frac{4\mu^2 e^2}{3\hbar^2}|P_{nm}|^2 A. \tag{84}$$

[12]The exclusion of central impacts means that we must preliminarily give up the (ability to?) interpret an extremely interesting group of phenomena, namely the transparency of atoms to slow electrons (Ramsauer Effect).

[13]S.E. Jahnke and F. Emde, Funktionentafeln, Leipzig 1909, S. 19.

We want to briefly discuss this expression for the yield function.

First one sees that in our approximation, the yield is proportional to $|P_{nm}|^2$, ie: the yield is proportional to the co-efficients of the transition probabilities b_{nm} of the Einstein radiation theory. These co-efficients correspond to the processes of absorption and stimulated emission (but no the probabilities for spontaneous emission $a_{nm} = \frac{8\pi h\nu_{nm}^3}{c^3} b_{nm}$)[14].

The yield for elastic reflection is proportional to $|P_{nn}|^2$, a magnitude which is optically unfeasible. In general, the diagonal elements of the matrix P_{nm} become zero[15], except in a few cases where the linear Stark effect exists (such as the hydrogen atom). Herr Pauli has informed me that he can even deduce that the diagonal elements of quadrupole and higher multipole moments vanish for the s-wave states of the alkali metals and for the ground states of the noble gases and alkali-earth metals. This result represents an exact expression for the spherical symmetry of the effective [scattering?] region of the atom. Our approximation is insufficient for the calculation of elastic reflections. For this, one must take the approximation a step further. It should soon be possible to test our theory against large quantities of data (Lenard and others) of mean free paths of electons in unexcited gases. Without exact calculation one can see that terms of fourth order in P_{nm} will determine the yield. These terms are much smaller than $|P_{nm}|^2$. Thereafter we can see that the cross-section of atoms for slow electrons (which is of the same order of magnitude as that calculated using the kinetic theory of gases) is far smaller than the cross-section for fast electrons (which are capable of exciting the atom)[16].

The dependence of the yield on direction is determined by the function A^2 from (82). This apparently corresponds to diffraction.

W. Elsasser[17] drew this conclusion from the de Broglie theory about a year ago. In taking seriously the wave picture, he concluded that slow electrons must scatter off an atom in such a way that their distribution after scattering should correspond approximately to the intensity pattern of light diffracting around a small sphere[18]. He showed the connection between the observations of Ramsauer[19] about the mean free path of elec-

[14]S.J.H. van Vleck, Phys. Rev. **23**, 330, 1924; Journ. Opt. Soc. Amer. **9**, 27, 1924. M. Born and P. Jordan, ZS. f. Phys. **33**, 479, 1925.

[15]In the case of the harmonic oscillator for example, they are zero, in the case of the anharmonic oscillator, they exist.

[16]This may be found in the literature in the recently published book by J. Franck and P. Jordan, "Anregung von Quantensprüngen durch Stösse" (Berlin, J. Springer, 1926).

[17]W. Elsasser, Die Naturwiss. **13**, 711, 1925. The relations between orders of magnitude upon which Elsasser's considerations are based, are based on de Broglie's formula for the wavelength:

$$\lambda = \frac{2\pi}{k} = \frac{h}{\sqrt{2\mu W}}$$

For $300V$ radiation, one has roughtly $\lambda = 7 \times 10^{-9} cm$, in other words, waves of atomic dimensions.

[18]S.K. Schwartzchild, Sitzungsber. d. Kgl. Bayer. Akad. d. Wiss., S. 293, 1901; G. Mie, Ann d. Phys. **25**, 377, 1908; P. Debye, Ann. d. Phys. **30**, 57, 1909.

[19]C. Ramsauer, Ann. d. Phys. **64**, 513, 1921; **66**, 546, 1921; **72**, 345, 1923. For further literature see "Ergebnisse der exakten Naturwissenschaften", 3. Bd. (Berlin, J. Springer, 1924), Artikel R. Minkowski

trons and the experiments of Davisson and Kunsman[20] about the angular distribution of electrons being reflected off a platinum plate. In the mean time, the correctness of these considerations has been proven by the experiments of Dymond[21]. He directly observed the occurance of interference maxima of electrons reflected off helium. A proof of our formalism from the data will follow later.

10 Final Remarks

Based on the previous considerations, I would like to express the opinion that Quantum Mechanics not only permits the formulation and solution of the problem of stationary states, but also the problem of transition processes. The Schrödinger formulation seems to do justice to these problems in the simplest way. Furthermore, it allows us to retain our conventional [ordinary?] view of space and time, in which events take place in a wholly normal manner. However, the proposed theory is not in accordance with the consequences of the causal determinism of single events. I have especially stressed this point about indeterminism in my preliminary publication, since it seems to me to be in agreement with the practical experience of experimenters. It is however natural for anyone who is dissatisfied with the above interpretation, to freely assume the existence of further parameters which may be introduced into the theory and which will determine single events. In Classical Mechanics, this appears as the "phase" of motion. For example, the coördinates of a particle at a certain instant. At first it seemed to me improbable that one could include physical quantities corresponding to these phases into the new theory. However, Herr Frenkel as informed me that it may be possible after all. In any case, this possibility would not change the practical indeterminism of collision processes, since one cannot give the values of these phases. In addition, this possibility must lead to the same formulas which appear in the suggested "phaseless" theory.

I would like to believe that the laws of motion for light quanta allow themselves to be treated in a completely analogous manner[22]. Now, immediately with the basic problem of free radiation, we no longer have a periodic process, but rather a decay process. This means an initial value problem rather than a boundary value problem for the for the coupled wave equation of Schrödinger's ψ and the electromagnetic field. Understanding the laws of this coupling is probably one of the most urgent problems. This is, as I know, being worked on in more than one place[23]. When these laws have been formulated, it will perhaps become possible to develop: a rational theory of lifetimes of states, the transition probabilities of radiative processes and the damping and spectral line widths.

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and H. Sponer, S. 67.

[20] Davisson and Kunsman, Phys. Rev. **22**, 243, 1923.

[21] Dymond, Nature.

[22] The difficulties one has encountered so far with the introduction of "ghost fields" into optics seems to me to be based partially on the implicit assumption that the wave centre and emitting particle must be in the same place. This is however already not the case with the Compton effect and will probably never be the case in general.

[23] See for example the recently appearing publication by O. Klein, ZS. f. Phys. **37**, 895, 1926.

11 Translators' Notes and Acknowledgments

I would like to thank Andreas Zech and Michael Sindel, without whose valued assistance, this translation would have been impossible.

The purpose of this translation is to analyse the language and analogy used by Born. Hence some of the translation is rather literal and if there is some uncertainty in his language use, I have put alternatives in square parenthesis. If I am still unsure of the best translation, the alternative is accompanied by a question mark. Otherwise the alternatives may have a more elaborative purpose. There are occasions where the exact meaning is not important and in these places I have tended to be loose in translation. I have also taken the liberty of updating some, but not all of the antiquated notation.