

Energy conservation laws in classical electrodynamics

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There are three electromagnetic integrals of motion that can be interpreted as the energy. These are the background energy, the elastic energy and the integral in the torsion field commonly referred to as the energy of the electromagnetic field. The integral in the torsion field gains the meaning of the energy inasmuch as it is concerned with the mechanical energy of a charged particle.

1. INTRODUCTION. THE ELECTROMAGNETIC FIELD

We will consider equations for electromagnetic potentials

$$\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} + \nabla \varphi = 0, \quad (1)$$

$$\nabla \cdot \mathbf{E} = 4\pi q \delta(\mathbf{x} - \mathbf{x}'), \quad (2)$$

$$\frac{\partial \mathbf{E}}{\partial t} - c \nabla \times (\nabla \times \mathbf{A}) + 4\pi q \mathbf{v} \delta(\mathbf{x} - \mathbf{x}') = 0. \quad (3)$$

The consideration will be restricted to the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0. \quad (4)$$

We will find three integrals of equations (1) – (4) that can be interpreted as the energy. This enables us to elucidate the concept of the energy of the electromagnetic field. Summation over recurrent index is implied throughout.

2. THE BACKGROUND ENERGY

Following [1] we express the vector field \mathbf{E} via some tensor field η_{ik} :

$$E_i = \kappa \frac{\partial \eta_{ik}}{\partial x_k}, \quad (5)$$

where κ is an arbitrary constant. Then (3) can be obtained convolving equation

$$\kappa \frac{\partial \eta_{ik}}{\partial t} + c \left(\frac{\partial A_i}{\partial x_k} + \frac{\partial A_k}{\partial x_i} \right) - q v_i \frac{\partial}{\partial x_k} \frac{1}{|\mathbf{x} - \mathbf{x}'|} = 0. \quad (6)$$

In derivation of (3) from (6) we used (5), (4) and following relations

$$\nabla(\nabla \cdot) = \nabla^2 + \nabla \times (\nabla \times), \quad (7)$$

$$\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi \delta(\mathbf{x} - \mathbf{x}'). \quad (8)$$

Taking in (6) $i = k$ and summing over the repeated index we get with the account of (4)

$$\kappa \frac{\partial \eta_{kk}}{\partial t} - q v_k \frac{\partial}{\partial x_k} \frac{1}{|\mathbf{x} - \mathbf{x}'|} = 0. \quad (9)$$

Integrating (9) all over the space

$$\frac{\partial}{\partial t} \int \eta_{kk} d^3x = 0. \quad (10)$$

The quantity

$$\frac{1}{2} \eta_{kk} \quad (11)$$

is interpreted in a mechanical model [1] as the density of the background energy of a substratum.

3. THE ELASTIC ENERGY

Let us define the displacement field \mathbf{s} by

$$\mathbf{A} = \kappa c \frac{\partial \mathbf{s}}{\partial t}. \quad (12)$$

Consider the case $\mathbf{v} = 0$. Substituting (12) into (3) and integrating it over time

$$\mathbf{E} = \kappa c^2 \nabla \times \nabla \times \mathbf{s} + \mathbf{h}(\mathbf{x}). \quad (13)$$

Substituting (12) and (13) into (1) we have by virtue of (2) and (4)

$$\frac{\partial^2 \mathbf{s}}{\partial t^2} + c^2 \nabla \times \nabla \times \mathbf{s} = 0, \quad (14)$$

$$\mathbf{h} + \nabla \varphi = 0. \quad (15)$$

Multiplying (14) by $\partial \mathbf{s} / \partial t$, integrating over the space and taking the second integral by parts we get

$$\frac{\partial}{\partial t} \frac{1}{2} \int [(\frac{\partial \mathbf{s}}{\partial t})^2 + (c \nabla \times \mathbf{s})^2] d^3 x = 0. \quad (16)$$

This integral of motion is interpreted in the mechanical analogy [2] as the elastic energy of a substratum.

4. CONSERVATION IN THE TORSION FIELD

Taking the curl of (14)

$$\frac{\partial^2 (\nabla \times \mathbf{s})}{\partial t^2} + c^2 \nabla \times (\nabla \times \nabla \times \mathbf{s}) = 0. \quad (17)$$

Multiplying (17) by $\partial (\nabla \times \mathbf{s}) / \partial t$, integrating over the space and taking the second integral by parts we get

$$\frac{\partial}{\partial t} \frac{1}{2} \int [(\nabla \times \frac{\partial \mathbf{s}}{\partial t})^2 + (c \nabla \times \nabla \times \mathbf{s})^2] d^3 x = 0. \quad (18)$$

Substituting (14) and then (12) into expression (18) we convert it into the electromagnetic form

$$\frac{\partial}{\partial t} \frac{1}{2} \int [(\nabla \times \mathbf{A})^2 + (\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t})^2] d^3 x = 0. \quad (19)$$

5. THE ELECTROMAGNETIC ENERGY

We will consider two charged particles. Forms (2) and (3) are specified by

$$\nabla \cdot \mathbf{E} = 4\pi q_1 \delta(\mathbf{x} - \mathbf{x}^{(1)}) + 4\pi q_2 \delta(\mathbf{x} - \mathbf{x}^{(2)}), \quad (20)$$

$$\frac{\partial \mathbf{E}}{\partial t} - c \nabla \times (\nabla \times \mathbf{A}) + 4\pi q_1 \mathbf{v}^{(1)} \delta(\mathbf{x} - \mathbf{x}^{(1)}) + 4\pi q_2 \mathbf{v}^{(2)} \delta(\mathbf{x} - \mathbf{x}^{(2)}) = 0. \quad (21)$$

The motion of the particles can be described by equations

$$\mathbf{v}^{(1)} = \frac{d\mathbf{x}^{(1)}}{dt}, \quad m_1 \frac{d\mathbf{v}^{(1)}}{dt} = q_1 \mathbf{E}(\mathbf{x}^{(1)}) + q_1 \frac{\mathbf{v}^{(1)}}{c} \times \nabla \times \mathbf{A}(\mathbf{x}^{(1)}), \quad (22)$$

$$\mathbf{v}^{(2)} = \frac{d\mathbf{x}^{(2)}}{dt}, \quad m_2 \frac{d\mathbf{v}^{(2)}}{dt} = q_2 \mathbf{E}(\mathbf{x}^{(2)}) + q_2 \frac{\mathbf{v}^{(2)}}{c} \times \nabla \times \mathbf{A}(\mathbf{x}^{(2)}). \quad (23)$$

Equations (22) and (23) close up the set of Maxwell's equations (1), (20), (21) and (4).

Let us derive an integral of motion that is concerned with the mechanical energy of the particles. Multiply (21) by \mathbf{E} , then substitute (1) into the second term. Integrate all over the space and take the second integral by parts. This gives

$$\frac{\partial}{\partial t} \frac{1}{8\pi} \int [\mathbf{E}^2 + (\nabla \times \mathbf{A})^2] d^3x + q_1 \mathbf{v}^{(1)} \cdot \mathbf{E}(\mathbf{x}^{(1)}) + q_2 \mathbf{v}^{(2)} \cdot \mathbf{E}(\mathbf{x}^{(2)}) = 0. \quad (24)$$

Substitute (22) and (23) into (24). Also for our convenience we use (1) in the first term of the expression under the integral. Thus we get

$$\frac{\partial}{\partial t} \left\{ \frac{1}{8\pi} \int \left[\left(\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right)^2 + (\nabla \times \mathbf{A})^2 \right] d^3x + \frac{1}{2} m_1 \mathbf{v}^{(1)} \cdot \mathbf{v}^{(1)} + \frac{1}{2} m_2 \mathbf{v}^{(2)} \cdot \mathbf{v}^{(2)} \right\} = 0. \quad (25)$$

The first term in (25) is commonly interpreted as the energy of the electromagnetic field. However, comparing (25) with (19) we see that (25) generalizes the integral of motion in the torsion field of the displacement.

6. CONCLUSION

The electromagnetic untracation has no relation to the background energy nor to the elastic energy. It is concerned with a conservation law in the torsion field of a substratum.

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- [1] O.V.Troshkin, On wave properties of an incompressible fluid, *Physica A*, **168**, No 2, 881-898 (1990);
 O.V.Troshkin, Perturbation waves in turbulent media, *Comp. Maths Math. Phys.*, **33**, No 12, 1613-1628 (1993).
 [2] V.P. Dmitriyev, Electrodynamics and elasticity, *Am.J.Phys.* **71**, No 9, 952-953 (2003).