GEOPHYSICS, VOL. 35, NO. 1 (FEBRUARY 1970), P. 153–157, 3 FIGS., 2 TABLES

SHORT NOTE

THE ANOMALOUS VERTICAL GRADIENT OF GRAVITY†

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The behavior of the earth's vertical gradient of gravity as observed, for example, in tall buildings or boreholes is a matter of considerable geophysical importance. The following elementary considerations may be of interest.

The theoretical "normal" vertical gradient of gravity varies only slightly with location and elevation on the earth's surface. Derivation from basic principles of potential theory and known data for the earth gives the well-known formula for the normal "free-air" gravity gradient (Hammer, 1938)

$$\frac{\partial g}{\partial h} = 0.308550 + 0.000227 \cos 2\phi - 0.145 \times 10^{-6}h, \tag{1}$$

in milligals per meter, where ϕ is the geocentric latitude (the small difference between geocentric and geographic latitude is negligible in this context) and h is the elevation in meters above sea level of the point of observation. The sign convention has been adopted to be positive in the sense of increasing gravity. In English units the vertical gradient is, in milligals per foot,

$$\frac{\partial g}{\partial h} = 0.09406 + 0.0000692 \cos 2\phi - 0.135 \times 10^{-7}h, \tag{2}$$

where h is in feet. In what follows, the *normal* free-air vertical gradient of equation (1) and equation (2) will be designated F.

The total change from equator to pole in the value of the normal free-air vertical gradient at sea level is less than two-tenths of one percent as shown in Table 1.

Table 1. Normal "free-air" vertical gradient of gravity at sea level

Latitude	mgal/m	mgal/ft
0°	0.3088	0.09413
45°	0.3086	0.09406
90°	0.3083	0.09399

The change with elevation amounts to about 0.05 percent per kilometer and 0.01 percent per 1000 ft. For all practical purposes the *normal* vertical gradient of gravity for the earth as a whole can be taken to be constant over the entire earth's surface (excluding only very high mountains), namely 0.3086 mgal/m or 0.09406 mgal/ft.

The variability caused by local gravity anomalies due to nonhomogeneity of rock density in the earth's crust and below is another matter. This may be defined as the "anomalous vertical gradient." Values of gradient anomaly in the few measurements which have been published range up to about ±5 percent, more than an order of magnitude larger than the variability in the normal gradient (Hammer, 1938; Thyssen-Bornemisza and Stackler, 1956; Kumagai et al, 1960; Kuo et al, 1969).[1] This is an important

[1] Recent unpublished data (Personal communication from Professor Charles Drake, Columbia University, New York) show a vertical gradient anomaly of $-17\frac{1}{2}$ percent in the eastern Mediterranean. This is two orders of magnitude larger than the range in values of the normal gradient.

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factor in using the normal vertical gradient value, for example in the calibration of gravimeters and the reduction of gravity station data in mountainous country. Analysis of the behavior of the anomalous vertical gradient by classical gravitational potential theory is straightforward but tedious (Morelli and Carrozzo, 1963). However, a very simple model calculation serves to illustrate the general magnitude and behavior.

Related analyses of the anomalous vertical gradient on the axis of a vertical cylinder, have been reported (Thyssen-Bornemisza, 1965; Elkins, 1966). The present note considers the problem from a different point of view.

The maximum variability in the vertical gradient which can occur is near a point mass. Let us therefore consider the vertical gradient of a local anomalous mass represented by a sphere with its top at the surface of the ground. The derivation is as follows.

The vertical component of the gravity anomaly of a spherical mass centered at depth z, at an elevation h above ground, and radial distance ρ from 0, shown in Figure 1B, is

$$A = GM(z+h)/[\rho^2 + (z+h)^2]^{3/2}, \quad (3)$$

where M = mass anomaly, and G is the gravitational constant. On the ground surface, the central magnitude of the anomaly (for $h=0, \rho=0$) is

$$A_0 = GM/z^2. \quad (4)$$

Limiting ourselves to the vertical, axial profile, $\rho=0$, we get

$$A = A_0 z^2/(z+h)^2. \quad (5)$$

The value of gravity in free air directly above the anomalous sphere, at elevation h, from the combined effects of the normal earth and the sphere is

$$g = g_0 - Fh + A_0 z^2/(z+h)^2, \quad (6)$$

where g_0 is the value of gravity at point 0 in Figure 1. Differentiating with respect to h, in the adopted sign convention,

$$\frac{\partial g}{\partial h} = F + 2A_0 z^2/(z+h)^3 \quad (7)$$

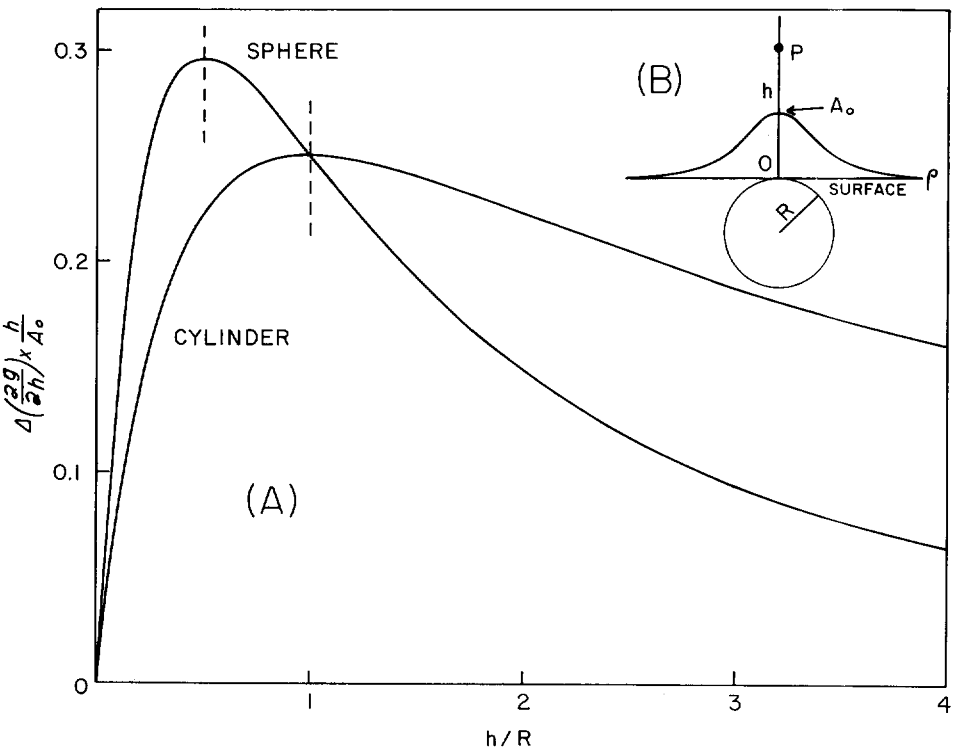

FIG. 1. (A) Dimensionless plot of vertical gradient, (B) Indicated gravity anomaly of spherical and horizontal cylindrical masses with tops at surface. The point of observation (P) of the vertical gradient is at elevation h above ground. The gravity anomaly is on the surface at elevation $h=0$.

gives the combined free-air vertical gradient along a vertical profile directly above the spherical mass.

The anomalous vertical gradient, defined as the departure from the theoretical normal value F, is

$$\Delta\left(\frac{\partial g}{\partial h}\right) = \frac{\partial g}{\partial h} - F = 2A_0z^2/(z+h)^3. \quad (8)$$

This shows the direct relationship on the anomaly axis between the sign of the anomalous vertical gradient and the associated gravity anomaly. Also, to the extent that a gravity anomaly may be approximated by that for a single sphere, equation (8) gives the quantitative magnitudes. Of course, on the flanks of an anomaly this simple relationship, and even the sign, will be different.

To proceed, assume that the top of the anomalous sphere is at the ground surface by taking $R=z$ in the equations. This will give an approximate simulation of the maximum gradient effect which can occur for a given gravity anomaly. Equation (8) then becomes

$$\Delta\left(\frac{\partial g}{\partial h}\right) = 2A_0R^2/(R+h)^3. \quad (9)$$

A dimensionless form is

$$\frac{h}{A_0}\cdot\Delta\left(\frac{\partial g}{\partial h}\right) = \frac{2h/R}{[1+(h/R)]^3}, \quad (10)$$

which is plotted in Figure 1A.

A similar analysis for an infinite horizontal cylinder with top at the surface gives the result

$$\frac{h}{A_0}\cdot\Delta\left(\frac{\partial g}{\partial h}\right) = \frac{h/R}{[1+(h/R)]^2}, \quad (11)$$

which is also plotted for comparison in Figure 1A. Aside from quantitative differences, the two models give essentially similar results. Note that both curves exhibit maxima. The significance of this fact will be discussed below.

The relationship of the vertical gradient and the density contrast (σ) is given by expressing the gravity anomaly A_0 in terms of radius and mass. The result for the sphere with top at the ground surface is

$$\Delta\left(\frac{\partial g}{\partial h}\right) = 8\pi G\sigma/3[1+(h/R)]^3, \quad (12)$$

and for the cylinder

$$\Delta\left(\frac{\partial g}{\partial h}\right) = 2\pi G\sigma/[1+(h/R)]^2. \quad (13)$$

The density contrast of the anomalous mass in the postulated models is uniquely determined in terms of assumed h/R and the magnitude of the anomalous vertical gradient.

A hypothetical application of the theory developed above follows. Assume that the observed value of the vertical gradient at a point 100 ft (30.5 m) above ground was 0.0950 mgal/ft (0.3117 mgal/m). This is an anomaly of +1%. Assume further that disturbing effects of building masses and terrain are negligible or have been accurately corrected. To analyze this result let us postulate that the gradient anomaly is caused by a spherical mass with its top at the ground surface directly beneath the observation point. The "observed" data give $h\Delta(\partial g/\partial h) = 100\times 0.00094 = 0.094$ mgal. (The same value results for the data expressed in metric units.) This value with the assumed values of the parameter h/R applied to the curve in Figure 1 yield the results listed in Table 2.

Table 2. Interpretation of gradient anomaly

	h/R	R (ft)	A_0 (mgal)	σ
Case a	2	50	0.634	+1.42
Case b	1	100	0.376	+0.442
Case c	$\frac{1}{2}$	200	0.317	+0.186
Case d	$\frac{1}{10}$	1000	0.626	+0.074

Case c, with $h/R=\frac{1}{2}$, is the extreme case. It indicates the smallest gravity anomaly that can account for the observed anomalous vertical gradient at the given elevation. The corresponding minimum gravity anomaly for the case of the horizontal cylinder ($h/R=1$) is 0.376 mgal, which is about 20 percent larger.

Discrimination between the several postulated cases require additional observations. One way is to make a horizontal survey of the area to define the gravity anomaly A, as indicated in Figure 2B. A second procedure (applicable in tall buildings, mine shafts, boreholes and—not inconceivably with developing gradiometer technology—in the air) is to extend the vertical gradient measurement to define its behavior over a range of elevations. The curves in Figure 2A show these

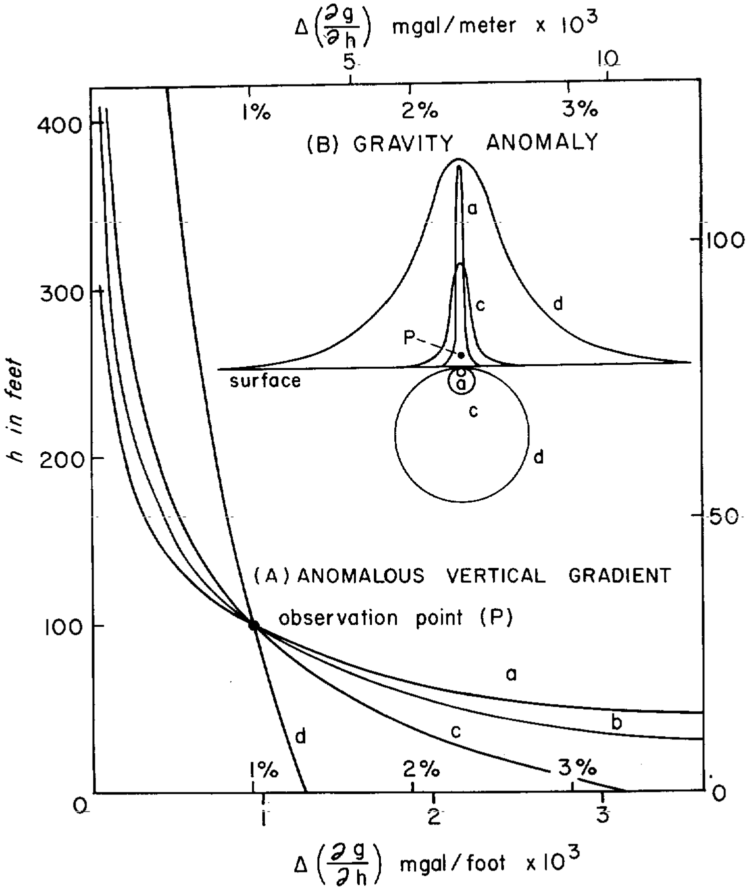

FIG. 2. (A) Anomalous vertical gradient, $\Delta(\partial g/\partial h)$, versus elevation h for postulated spherical masses with tops at surface. (B) Gravity anomalies of the masses on the surface $h=0$. Data refer to Table 2. (Case b is intermediate between cases c and d and is not plotted)

results. The vertical profiles of the anomalous vertical gradient are strongly diagnostic.

Interpreting the postulated gradient anomaly in terms of an infinitely-long horizontal cylinder with top at surface gives the curves in Figure 3. The results and conclusions are similar to that for the sphere.

SUMMARY

The model study reported in this paper is easily extended to single masses of other geometrical forms and depth. In such cases, "R" is any characteristic dimension (size or depth) of the model. For an assumed model and a given value of the axial anomalous vertical gradient at a known elevation, the magnitude of the associated gravity anomaly and a complete and unique interpretation (both R and σ) are derivable for any assumed value of the dimensionless ratio h/R.

The relationships between vertical gradient and areal gravity are easy to understand from basic principles. Minor exceptions which may occur (Kumagai et al, 1960) do not apply to localized features. Isolated anomalies in the vertical gradient *must* correlate directly with associated gravity anomalies. If they do not the data in one or the other or both are inadequate.

Nonlinear behavior of a vertical profile of the vertical gradient can occur only near localized (shallow) mass anomalies. Vertical gradient effects of strong, broad gravity anomalies tend to be small and vertically linear.

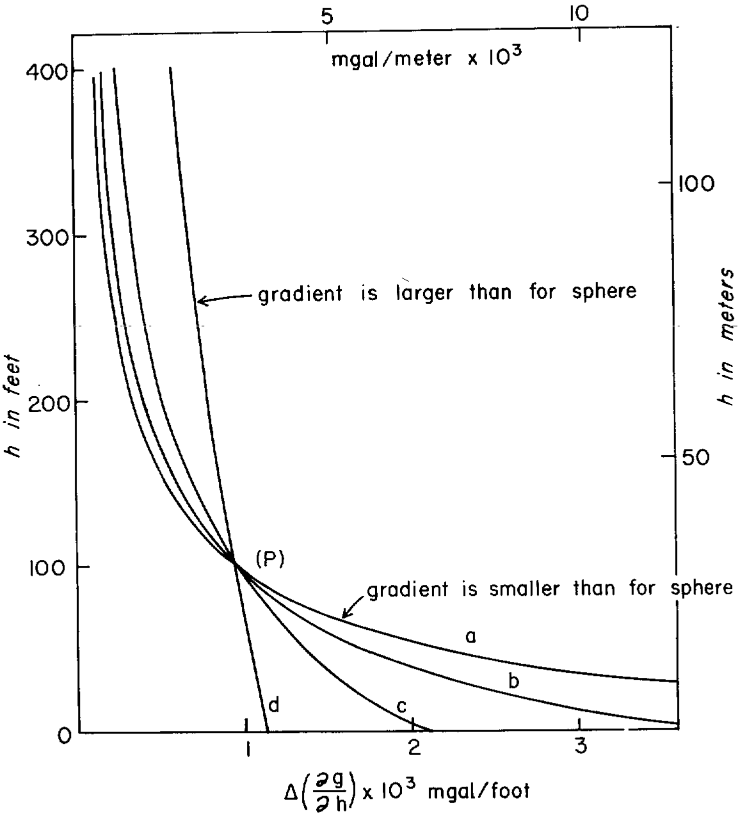

FIG. 3. Anomalous vertical gradient versus elevation for postulated infinite horizontal cylinders with tops at surface. Assumed parameters (h/R) for cases a, b, c, d, are respectively 2, 1, $\frac{1}{2}$, 1/10.

Vertical gradient measurements in tall buildings (and also in underground mine shafts and boreholes) should be supplemented with an areal gravity survey to define the locally anomalous gravity field in the vicinity. To check the reality of nonlinear vertical gradient effects by an areal gravity survey it is good practice to have the horizontal station spacing in the immediate vicinity closer than the nonlinear elevation intervals in the gradient data.

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