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A Non-Relativistic Look at the Compton Effect

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In a usual modern physics class the Compton effect is used as the pedagogical model for introducing relativity into quantum effects. The shift in photon wavelengths is usually introduced and derived using special relativity. Indeed, this works well for explaining the effect. However, in the senior author's class one of the student coauthors of this paper, Sandeep Giri, asked what would happen if classical expressions for the electrons' momentum ($m\mathbf{v}$) and kinetic energy [$(1/2)mv^2$] were used. The first response given to the question was that the relevant energies were relativistic and hence this approach would not work. Further thought led to the realization that the electron receives only the difference in the energies of the incoming and outgoing photons. This left the initial conclusion in doubt and we began a serious look at what would the answer be. As a result of our analyses, we believe that the Compton effect provides the clearest pedagogical test for the need of relativity in the case of gamma ray scattering while allowing both the classical and relativistic results to explain the original x-ray results of Compton.

In 1922 and 1923 Arthur Compton announced his famous wavelength shift for the case in which x-ray or gamma ray photons elastically scatter off of electrons.[1] This achievement was followed closely and independently by Peter Debye.[2] The shift in wavelength, $\Delta\lambda = \lambda_2 - \lambda_1$, was given by:

$$\Delta\lambda = \lambda_c(1 - \cos\varphi), \qquad (1)$$

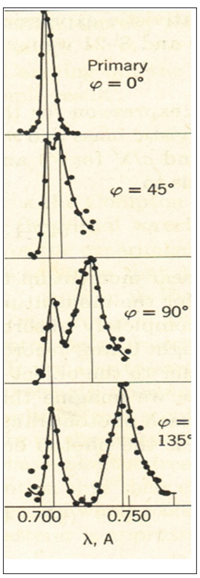

Fig. 1. Compton's original x-ray scattering data.[3,4] The ordinate is x-ray intensity, the abscissa is wavelength in Å.

where λ_c is the Compton wavelength, $h/mc = 0.0243$ Å, and φ is the angle between the scattered photon (λ_2) and the incoming photon (λ_1). It is apparent that the wavelength shift is independent of λ_1. In his announcement of the effect, Compton displayed data obtained using molybdenum K_α x-rays scattered off of graphite with a λ_1 of 0.711 Å. We present in Fig. 1 Compton's original data.[3,4] Compton found superb experimental agreement with Eq. (1); for example, $\Delta\lambda$ was found to be 0.0242 Å for 90° scattering, whereas Eq. (1) predicts 0.0243 Å .

Compton's derivation of Eq. (1) invoked a relativistic electron as might be expected from the use of x-ray or gamma ray energies. We provide in the appendix a derivation of this equation. Bartlett, in performing a thorough historical review of the Compton effect, mentioned that qualitative observation of longer wavelength secondary emissions was first observed in gamma ray experiments.[5] Compton was well aware of this result.[6,7] However, accurate quantitative work developed first for x-rays through the use of the Bragg spectrometer[5] and hence the quantitative verification of Eq. (1) used x-rays.

However, the electron may be non-relativistic since it acquires the energy difference between the incoming and outgoing photons. Thus, we decided to examine what Newtonian physics and the quantum assumption for the photons' momentum (p_{photon}) of

$$p_{photon} = h/\lambda \qquad (2)$$

would yield for the Compton effect. Remarkably, these simple assumptions work extremely well over large wavelength regimes. Also, we discuss new features seen in the non-relativistic theory and not in the relativistic prediction. We also provide some experimental results and comparisons between the usually reported (and correct) relativistic results and those derived here.

The non-relativistic Compton effect

Using the classical expressions for kinetic energy of the electron, $(1/2)mv^2$, and momentum vector of the electron, $m\mathbf{v}$, as well as Eq. (2) results in the non-relativistic analog to Eq. (1):

$$\Delta\lambda = \lambda_c(\lambda_2/2\lambda_1 + \lambda_1/2\lambda_2 - \cos\varphi). \qquad (3)$$

The appendix provides the derivation of this equation. Thus, the non-relativistic shift is wavelength *dependent*, a result Compton alluded to when he wrote:

> "It is perhaps surprising that the increase (in the scattered wavelength) should be the same for all wavelengths. Yet as a result of an extensive experimental study of the change in wave-length on scattering, the writer has concluded that 'over the range of primary rays from 0.7 to 0.025 Å, the wavelength of the secondary X-rays at 90° with the incident beam is roughly 0.03 Å greater than that of the primary beam which excites it.'[6]"[7]

However, for Compton's x-ray results given in Fig. 1, published widely today in numerous books on quantum physics and modern physics, the non-relativistic predictions [Eq. (3)] are fully in accord with experiment and cannot be distin-

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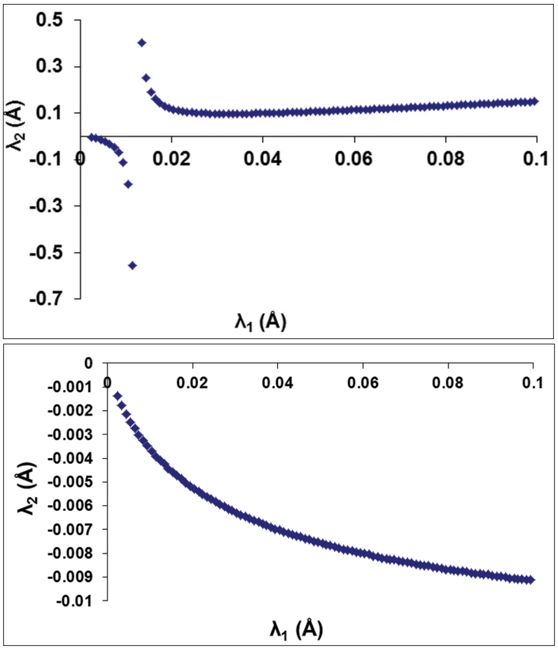

Fig. 2. (a) λ_2 as a function of λ_1 given by one solution of Eq. (5). The case shown is for 180° scattering and the solution contains some positive values. (b) λ_2 as a function of λ_1 given by another solution of Eq. (5). The case shown is for 180° scattering and the solution is only negative.

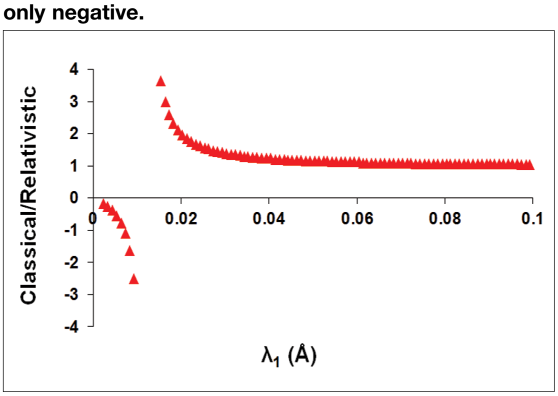

Fig. 3. Ratio of the non-relativistic Compton shift to the relativistic Compton shift as a function of λ_1 for the solution with positive roots. The case shown is 180° scattering.

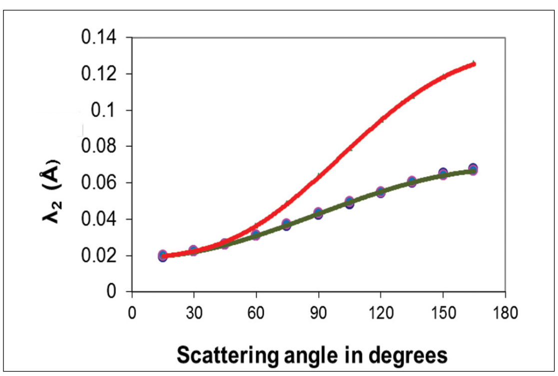

Fig. 4. Scattered wavelength (λ_2) as a function of scattering angle for ^{137}Cs. The upper curve is the non-relativistic prediction; the lower curve is the relativistic prediction. The data from French[8] are depicted by • .

guished from the predictions of Eq. (1) within the experimental error! This becomes clear since the photon wavelengths (λ_2 and λ_1) are so close to each other for all scattering angles and the first two terms in Eq. (3) thus reduce to near unity. Thus, Eq. (1) is equivalent to Eq. (3) for this case. In the most extreme case of backscattering (180°):

$$\lambda_1 = 0.711 \text{ Å} \tag{4a}$$

and

$$\lambda_2 = 0.760 \text{ Å}. \tag{4b}$$

Thus, the first two terms of Eq. (3) sum to 1.0023 and Eq. (3) differs from the relativistic Eq. (1) by just 0.115%.

Equation (3) can be rewritten so that λ_2 is an explicit function of λ_1 and φ. The result is a quadratic equation whose solutions are:

$$\lambda_2 = [\lambda_1 - \lambda_c \cos\varphi \pm \sqrt{((\lambda_1 - \lambda_c \cos\varphi)^2 + 2(1 - \lambda_c/2\lambda_1)\,\lambda_1\lambda_c)}]/\,2(1 - \lambda_c/2\lambda_1). \tag{5}$$

Equation (5) predicts two roots, only one of which has positive values. We depict in Figs. 2(a) and 2(b) these two solutions for λ_2 as a function of λ_1, for the backscattered condition. Notice that one root of λ_2 is always negative [as shown in Fig. 2(b)] and thus we do not consider it to have physical significance other than being a solution consistent with the classical conservation laws. The other root is positive for $\lambda_1 > \lambda_c/2$, while for values of $\lambda_1 < \lambda_c/2$ we see that λ_2 is negative. We also see that there is a divergence at $\lambda_c/2$ for this root. We conclude from this that the non-relativistic Compton effect may only occur for $\lambda_1 > \lambda_c/2$. As a further comparison of the relativistic Compton effect with the non-relativistic version, we present Fig. 3, which shows the ratio of the two Compton shifts, Eq. (3)/Eq. (1), as a function of λ_1 for the solution with positive values. We see that the significant deviation from unity occurs for $\lambda_c/2 < \lambda_1 < \lambda_c$. This implies that for this wavelength regime the electron receives enough energy to make the relativistic correction necessary. It is instructive to note that the divergence at $\lambda_1 = \lambda_c/2$ corresponds to complete momentum and energy transfer to the electron and none to an outgoing photon; in effect the incoming photon and electron experience a photoelectric effect.

Experimental tests of the relativistic and non-relativistic Compton shift formula

In 1965 Walter French described the results of very careful experiments on the Compton effect using gamma rays from ^{137}Cs as the source of the incoming photons.[8] His experimental results are depicted in Fig. 4, which also gives the relativistic and non-relativistic predictions from Eqs. (1) and (5), respectively. It is clear from inspection of this figure that the relativistic predictions give spectacular agreement with experiment for all scattering angles and that the classical predictions are clearly at variance with experiment for angles greater than 60°. These data are a much better test of the relativistic predictions versus the classical predictions than those of Compton, shown in Fig. 1, because of the use of a much shorter λ_1 (0.01873 Å versus 0.711 Å).

In our modern physics lab at Coe College during fall 2001, we performed a routine gamma ray spectroscopy lab in which energy spectra were obtained using a computerized multichannel analyzer (MCA) with a KI scintillation detector and PM-tube. Calibration of the MCA was provided by using the known gamma ray energies from ^{137}Cs, ^{60}Co, ^{22}Na, and ^{109}Cd. We then carefully examined the positions of the Compton backscattered peaks (180° scattering) for three easily measured isotopes. Table I provides these data along with the relativistic and classical predictions for the wavelength of the Compton scattered photons.

The Compton effect is a pedagogical delight since it provides such a clear test between classical and relativistic physics, unlike any other experiment at the level of modern physics. Thus, the three cases given in Table I provide simple and clearly distinguishable tests between relativistic and classical predictions. Again the relativistic predictions are in excellent accord with the experiment. Note that it is easy to enter the classical regime, $\lambda_1 < \lambda_c/2$, where no Compton shift should occur, thus providing a further clear and convincing comparison of the two theories. The case of ^{60}Co provides this test. Classically the Compton shift should not be possible—the prediction for λ_2 is negative. Yet, again, agreement is observed for the relativistic case.

Table I. Some Coe College modern physics lab data from fall 2001 on the Compton effect along with relativistic and classical predictions. The reported data are for Compton backscattering.

Wavelengths of the Compton shifted photons using incident gamma rays:				
Isotope	λ_1 (Å)	$\lambda_{2(\text{measured})}$ (Å)	$\lambda_{2(\text{relativistic})}$ (Å) Eq. (1)	$\lambda_{2(\text{classical})}$ (Å) Eq. (5)
^{22}Na	0.02426	0.070 (+/– 0.005)	0.073	0.103
^{137}Cs	0.01873	0.066 (+/– 0.005)	0.067	0.127
^{60}Co*	0.00990	0.056 (+/– 0.004)	0.059	–0.148

* ^{60}Co data are the average of two closely spaced gamma ray energies.

Conclusions

The non-relativistic Compton shift has been investigated. Using the classical relations for energy and momentum as well as the quantum expression for momentum of a photon, a shift in scattered wavelength was found for photon-electron collisions (the Compton effect). This was compared to the wavelength shift found by employing the relativistic energy-momentum relation. Unlike the relativistic solution, the non-relativistic solution contains two roots for the scattered wavelength, one of which is always negative. For the root with positive values of the scattered wavelength, an asymptote was found near $\lambda_c/2$. Below $\lambda_c/2$ the non-relativistic solution is negative. For photons with wavelength greater than the Compton wavelength λ_c, the relativistic and non-relativistic shifts are virtually identical. Experimental results agree with the relativistic predictions in all cases.

We think the Compton effect presents the clearest experimental example (the scattering of gamma rays off electrons) for the need for relativity in an introductory modern physics class and lab. Additionally, it has the pedagogical benefit of showing that in x-ray scattering the classically derived result may be used. The contrast between x-ray scattering and gamma ray scattering is striking.

The moral of this story is to listen very carefully to questions from students. They can lead to keen new insights into physics.

Acknowledgments

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References

1. A. H. Compton "A quantum theory of the scattering of x-ray by light elements," *Phys. Rev.* **21**, 207 (abstract) (Feb. 1923); A. H. Compton, "A quantum theory of the scattering of x-ray by light elements," *Phys. Rev.* **21**, 483–502 (1923); A. H. Compton, "Secondary radiations produced by x-rays," *Bull. Natl. Res. Council* **20**, 16–72 (1922).
2. P. Debye, "Zerstreuung von Rontgenstrahlen und Quantentheorie," *Physik. Z.* **24**, 161–166. (1923).
3. Compton's original data appear in A. H. Compton, "The spectrum of scattered x-rays," *Phys. Rev.* **22** (5), 409–413 (1923). The abscissa in the original paper was in terms of angle.
4. The original Compton data were replotted as a function of wavelength shift in Å in D. Halliday and R. Resnick, *Physics, Part II* (Wiley, New York, 1962), p. 1185.
5. A. A. Bartlett, "Compton effect: Historical background," *Am. J. Phys.* **32** (2), 120–142 (Feb. 1964).
6. A. H. Compton, "Secondary radiations produced by x-rays," *Bull. Natl. Res. Council* **20**, 16–72 (1922).
7. A. H. Compton, "A quantum theory of the scattering of x-ray by light elements," *Phys. Rev.* **21**, 483–502 (1923).
8. W. R. French Jr. "Precision Compton-effect experiment," *Am. J. Phys.* **33** (7), 523–527 (July 1965).

Appendix

We provide here derivations of both the relativistic and classical (non-relativistic) Compton effect shift formulas.

Relativistic Compton shift

We start with the momentum of the electron, $\mathbf{P}_e$, given by the difference between the photon momenta ($\mathbf{P}_1$, $\mathbf{P}_2$):

$$\mathbf{P}_e = \mathbf{P}_1 - \mathbf{P}_2 . \quad \text{(A1)}$$

Squaring both sides yields:

$$P_e^2 = P_1^2 + P_2^2 - 2P_1P_2 \cos\varphi. \tag{A2}$$

Conservation of relativistic energy produces:

$$P_1c + mc^2 = P_2c + \sqrt{[mc^2 + (P_ec)^2]}. \tag{A3}$$

Squaring Eq. (A3) and rearranging results in:

$$P_e^2 = P_1^2 + P_2^2 - 2P_1P_2 + 2mc(P_1 - P_2). \tag{A4}$$

Combining Eqs. (A2) and (A4) and use of the quantum result for the momentum of a photon, $P = h/\lambda$, produces the usual Compton shift:

$$\Delta\lambda = \lambda_c(1 - \cos\varphi), \tag{1}$$

where $\Delta\lambda = \lambda_2 - \lambda_1$.

Non-Relativistic

Equations (A1) and (A2) are used as in the relativistic case. The conservation of energy expression (classical version of A3) and use of electron momentum, $m\mathbf{v}$, results in Eq. (A3) changing to:

$$P_1c = P_2c + (0.5)mv^2 = P_2c + P_e^2/2m. \tag{A5}$$

Dividing Eq. (A2) by $2m$ allows P_e to be eliminated through the simultaneous use of Eq. (A2) and Eq. (A5). After application of the quantum expression for photon momenta, the non-relativistic Compton shift becomes:

$$\Delta\lambda = \lambda_c(\lambda_2/2\lambda_1 + \lambda_1/2\lambda_2 - \cos\varphi). \tag{3}$$

The logic of the two derivations is identical.

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