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Approximate Optimal Guidance for the Advanced Launch System

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Abstract

A real-time guidance scheme for the problem of maximizing the payload into orbit subject to the equations of motion for a rocket over a spherical, nonrotating Earth is presented. An approximate optimal launch guidance law is developed based upon an asymptotic expansion of the Hamilton-Jacobi-Bellman or dynamic programming equation. The expansion is performed in terms of a small parameter, which is used to separate the dynamics of the problem into primary and perturbation dynamics. For the zeroth-order problem the small parameter is set to zero and a closed-form solution to the zeroth-order expansion term of the Hamilton-Jacobi-Bellman equation is obtained. Higher-order terms of the expansion include the effects of the neglected perturbation dynamics. These higher-order terms are determined from the solution of first-order linear partial differential equations requiring only the evaluation of quadratures. This technique is preferred as a real-time on-line guidance scheme to alternative numerical iterative optimization schemes because of the unreliable convergence properties of these iterative guidance schemes and because the quadratures needed for the approximate optimal guidance law can be performed rapidly and by parallel processing. Even if the approximate solution is not nearly optimal, when using this technique the zeroth-order solution

always provides a path which satisfies the terminal constraints. Results for two-degree-of-freedom simulations are presented for the simplified problem of flight in the equatorial plane and compared to the guidance scheme generated by the shooting method which is an iterative second-order technique.

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English Symbols

a, b, c	constants of the quadratic mass equation
C_D	drag coefficient
C_{D_α}	linear coefficient in the drag model
$C_{D_{\alpha^2}}$	quadratic coefficient in the drag model
$C_{D_{\alpha^3}}$	cubic coefficient in the drag model
C_L	lift coefficient
C_{L_α}	linear coefficient in the lift model
$C_{L_{\alpha^2}}$	quadratic coefficient in the lift model
C_Q	side force coefficient
C_u, C_w	constant terms associated with the Lagrange multipliers for the velocity components u, w
$\overline{C}_w$	constant term used to rewrite the Lagrange multipliers in terms of mass, $\overline{C}_w = \frac{\lambda_h}{\sigma T} m_0 + C_w$
$\overline{C}_w'$	second stage value of $\overline{C}_w$ given first stage initial conditions
D	drag force
$f(y, u, \tau)$	primary dynamics
f_i	the ith term of the asymptotic expansion of the primary dynamics
f_u	partial derivative of the primary dynamics with respect to the control u

$g(y, u, \tau)$	perturbation or secondary dynamics
g_i	the ith term of the asymptotic expansion of the perturbation dynamics
g_u	partial derivative of the perturbation dynamics with respect to the control u
g	gravity
g_s	sea-level gravity
$G(y, u, t)$	scalar function of the augmented performance index
h	altitude
h_f	final attained altitude
$h_{f_{spec}}$	specified final altitude
h_s	atmospheric density scale height
H	the Hamiltonian of the system
H^{opt}	the optimal Hamiltonian
H_{wind}	the Hamiltonian of the wind axis system
H_{LH}	the Hamiltonian of the local horizon or Cartesian system
H_f	the Hamiltonian evaluated at the final time
H_u	first derivative of the Hamiltonian with respect to the control u
H_{uu}	second derivative of the Hamiltonian with respect to the control u
I_{sp}	specific impulse
J	performance index
$K(Q, P, t)$	Hamiltonian for a new set of variables Q and P
L	lift force

$L, \bar{L}$	Lagrangians used in Appendix B
m	mass of the vehicle
m_f	final mass
m_{stage_1}	specified mass at end of first stage before staging
m_{stage_2}	specified mass at beginning of second stage after staging
M	Mach number; $M = \frac{V}{sos}$
n	number of engines
$N(y,t)$	dynamic pressure equality constraint appears in Appendix C
N_x	the partial of the dynamic pressure equality constraint
p	generalized coordinate of old system in Appendix B
P	generalized coordinate of new system in Appendix B
$P(x,t)$	the optimal return function starting at the initial conditions
P_x	the partial derivative of the optimal return function with respect to the initial state x
P_t	the partial derivative of the optimal return function with respect to the initial time t
P_i	i^{th} term of the asymptotic expansion of the primary dynamics
P_{i_x}	the partial derivative of the ith term of the expansion of the optimal return function with respect to the initial state x
P_{i_t}	the partial derivative of the ith term of the expansion of the optimal return function with respect to the initial time t
q	dynamic pressure generalized coordinate of old system in Appendix B
Q	side force in Chapter 3 on ALS modelling generalized coordinate of new system in Appendix B

r	radial position of the vehicle: $r_e + h$
r_e	radius of the Earth
R_i	the forcing function associated with the ith correction term
sos	speed of sound
S	Cross-sectional area of the combined vehicle
$S(q, Q, t)$	generating function defined in Appendix B
t, t_0	initial time
t_f	final time
t_{stage}	stage time
T	total thrust of the vehicle
T_1	value of the thrust for the first stage
T_2	value of the thrust for the second stage
T_{vac}	vacuum thrust per engine
u_i	the ith term of the asymptotic expansion series of the control
(u, v, w)	velocity components associated with the inertial frame
V	velocity
V_f	final attained velocity
$V_{f_{spec}}$	specified final velocity
x	initial states
X	downrange
(X, Y, Z)	Position coordinates for the right-handed inertial frame
y	state vector

Greek Symbols

α	angle-of-attack; control in the wind axis system
β	vehicle sideslip angle; control in the wind axis system
χ	velocity heading angle
$\delta(\epsilon, h)$	ratio of the atmospheric density to the small parameter ϵ
Δ	discriminant associated with the quadratic mass equation
	$\Delta = 4ac - b^2$
Δm_{stage}	discontinuity in the mass at staging
ϵ	the small expansion parameter;
	ratio of the atmospheric scale height to the radius of the Earth
ϵ^j	the jth power of the small expansion parameter
γ	flight path angle
γ_f	final attained flight path angle
$\gamma_{f,pec}$	specified final flight path angle
λ_y	Lagrange multiplier associated with the state y
$\lambda_V, \lambda_\gamma, \lambda_\chi$	Lagrange multipliers associated with the wind axis states
$\lambda_h, \lambda_\theta, \lambda_\phi, \lambda_m$	
$\lambda_u, \lambda_v, \lambda_w$	Lagrange multipliers associated with the Cartesian states
$\lambda_h, \lambda_X, \lambda_Y, \lambda_m$	
μ	velocity roll angle; control in the wind axis system
ν_y	Lagrange multiplier associated with the terminal constraint on y
$\Omega(y(t_{stage}))$	constraint imposed by the staging condition of the rocket
ϕ	latitude
$\psi(q, p, t)$	new generating function equal to $S(q, Q, t)$
$\phi(y_f, \tau_f)$	scalar component of performance index

$\Psi(y_f)$	vector of terminal constraints
ρ	atmospheric density
ρ_s	sea-level atmospheric density
ρ_r	reference atmospheric density
σ	specific fuel consumption
τ	time
θ	longitude
θ_p	pitch angle; control in the Cartesian system

Miscellaneous Symbols

nm	nautical mile
sin	sine function
cos	cosine function
tan	tangent function
$sinh^{-1}$	inverse hyperbolic sine function
$\Im(m)$	argument of the inverse hyperbolic sine function
$d(\ )$	the differential of ( )
$\delta(\ )$	the time-varying variation of ( )
$\bar{\delta}(\ )$	the variation of ( ) with time held fixed
$\frac{d}{d\tau} = \dot{(\ )}$	denotes the time derivative of ( ) with respect to the independent variable time
$\frac{d}{dm}(\ )$	partial derivative of ( ) with respect to the independent variable mass
$\frac{\partial}{\partial x}(\ )$	partial derivative of ( ) with respect to the initial state x

$\frac{\partial}{\partial t}(\ )$ partial derivative of ( ) with respect to the initial time t

$(\ )'$ prime superscript used for second stage values which are linked to the initial conditions on the first stage subarc

$(\ )_0$ subscript denotes the initial conditon of ( )

$(\ )_f$ subscript denotes the final conditon of ( )

$(\ )^0$ superscript denotes the optimal ( )

$(\ )_s$ subscript denotes sea-level value;

            subscript denotes the characteristic direction in Chapter 2

lim   limit operation

Chapter 1

Introduction

An approach to real-time optimal launch guidance is suggested here based upon an expansion of the Hamilton-Jacobi-Bellman or dynamic programming equation. In the past, singular perturbation theory has been used in expansion techniques used to solve optimization problems [1, 2, 3]. For singular perturbation methods the states are split up into a set of 'fast' and 'slow' variables. The solution is then sought in two separate regions; one region where the fast states are dominant and an outer region where the slow states are determined. A composite solution can then be determined by combining the two solutions. Matching asymptotic expansions is one method for obtaining the final solution. This research uses a regular asymptotic expansion which is assumed valid over the entire trajectory of the launch optimization problem. An example of a launch optimal control problem is to determine the angle-of-attack profile which maximizes the payload into orbit subject to the dynamic constraints of a point mass model over a rotating spherical Earth. The solution of this type of optimization problem is obtained by an iterative optimization technique. Since the convergence rate of iterative techniques is difficult to quantify and convergence is difficult to prove, these schemes are not suggested to be used as the basis for an on-line real-time guidance law.

In contrast, an approximation approach is developed which is based

upon the physics of the problem. Thrust and gravity are assumed to be the dominant forces encountered by the rocket while the angle-of-attack is usually kept small in order to minimize the effect of the aerodynamic forces acting on the vehicle. Numerical optimization studies [4] have been performed which support this assumption. These results also indicate that ignoring the aerodynamic pitching moment has a negligible effect on the performance of the vehicle. Thus the launch problem would seem to lend itself to the use of perturbation theory. It is shown that the forces in the equations of motion can be written as the sum of the dominant forces and the perturbation forces which are multiplied by a small parameter ϵ, where ϵ is the ratio of the atmospheric scale height to the radius of the Earth. The motivation for this decomposition is that for $\epsilon = 0$, the problem of maximizing the payload into orbit subject to the dynamics of a rocket in a vacuum over a flat Earth, is an integrable optimal control problem. The perturbation forcing terms in the dynamics produce a nonintegrable optimal control problem. However, since these perturbation forces enter in with a small parameter, an expansion technique is suggested based upon the Hamilton-Jacobi-Bellman equation. The expansion is made about the zeroth-order solution determined when $\epsilon = 0$. This zeroth-order problem is now solved routinely in the generalized guidance law for the Space Shuttle [5] with a predictor/corrector scheme employed to guide the vehicle along the desired path.

The higher-order terms of the expansion are determined from the solution of first-order linear partial differential equations which require only integrations which are quadratures. Quadratures are integrals in which the integrand is only a function of the independent variable. Previous solution meth-

ods applied to guidance problems have motivated the approach suggested here. These include the explicit guidance laws, E-guidance, developed by George Cherry [6] for the Apollo flight. By writing the dynamics strictly as functions of the independent variable a solution was obtained by quadrature integrations. Past applications [7, 8] of the proposed scheme, have shown that very close agreement with the numerical optimal path is obtained by including only the first-order term. Because no iterative technique is required, this scheme is suggested as a guidance law since the quadratures can be performed rapidly.

Chapter 2 contains a general formulation of the perturbation problem associated with the Hamilton-Jacobi-Bellman partial differential equation (HJB-PDE). The technique for determining the higher-order expansion terms due to the perturbation forces caused by the atmosphere and the spherical Earth model is discussed. Lastly, the recursive relationship for the control is presented. In Chapter 3, the characteristics for the Advanced Launch System (aka National Launch System) and the general equations of motion in terms of the small parameter ϵ, are given. For $\epsilon = 0$, a simplified optimal launch problem in the equatorial plane is formulated, and its solution in terms of elementary functions is given in Chapter 4. The coordinate system transformation used to obtain the analytic solution is included. Also discussed is the linking of the trajectory subarc for the first stage to the subarc of the second stage. In Chapter 5 the first-order correction term to the control is determined. Results are presented in Chapter 6 and compared to the shooting method solution, which is a numerical iterative second-order optimization technique. It was found that during much of the first stage the aerodynamics are not small when flying the optimal vacuum trajectory. Chapter 7 presents a method for reshaping the

zeroth-order trajectory by including an aerodynamic effect. This effort centers on the use of constant aerodynamic pulse functions which are obtained by averaging the aerodynamics along the zeroth-order path during various time intervals. Lastly, Chapter 8 relates perturbation theory and the Calculus of Variations with the expansion of the Hamilton-Jacobi-Bellman equation. The equivalence of the two solution methods is presented.

Chapter 2

The Peturbed Hamilton-Jacobi-Bellman Equation

The optimal control problem can be formulated as one which minimizes a performance index subject to a set of nonlinear dynamics and a set of terminal constraints; that is,

Minimize

$$J = \phi(y_f, \tau_f) \tag{2.1}$$

with the dynamics

$$\dot{y} = f(y, u, \tau) + \epsilon g(y, u, \tau) \tag{2.2}$$

subject to the terminal constraints

$$\Psi(y_f, \tau_f) = 0 \tag{2.3}$$

and the initial conditions

$$y(t) = x = given \tag{2.4}$$

Note that y is an n-dimensional state vector, u is an m-dimensional control vector, ϵ is a small parameter, τ is the independent variable, $\dot{y} \triangleq dy/d\tau$, t is the initial value of the independent variable, and x is the initial state at t.

Eq. (2.2) is separated into two portions: primary and secondary dynamics. Note that the control appears in both parts. The primary dynamics

can be assumed to dominate over the secondary dynamics because the secondary dynamics are multiplied by the small parameter (ϵ) and therefore have a small perturbing effect on the system.

The Hamilton-Jacobi-Bellman (H-J-B) equation [9] is

$$- P_t = H^{opt} = \min_{u \in \mathcal{U}} H = P_x[f^{opt} + \epsilon g^{opt}] \tag{2.5}$$

where $\mathcal{U}$ is the class of piecewise continuous bounded controls and $u^{opt}(x, P_x, t)$ is obtained from the optimality condition $H_u = 0$ and from the assumption that the Legendre-Clebsch condition is satisfied (H_{uu} is positive definite). In addition, $f^{opt} \equiv f(x, u^{opt}, t)$ and $g^{opt} \equiv g(x, u^{opt}, t)$. The Hamilton-Jacobi-Bellman equation will be used to determine the optimal control policy which minimizes the cost criterion J.

The function $P(x, t)$ is called the optimal return function and is defined as the optimal value of the performance index for a path starting at x and t while satisfying the state equations (2.2) and the terminal constraints, i.e., $P(x, t) = \phi(y_f, \tau_f)$ at the hypersurface $\Psi(y_f, \tau_f) = 0$. The Hamilton-Jacobi-Bellman partial differential equation (2.5) can be interpreted [10] as the derivative of the optimal return function P. The optimal return function is a constant since it is dependent only on the terminal conditions and thus the total derivative of the optimal return function along an extremal path must be zero.

$$\frac{dP}{dt} = P_t + P_x[f^{opt} + \epsilon g^{opt}] = 0$$

Each point in space belonging to the optimal trajectory must give the same value to the optimal return function as the optimal $P(x, t)$ since the trajectory

is considered optimal from the initial conditions (x, t) to the terminal manifold. Now, if a non-optimal control is chosen at any point in the trajectory, then the resulting terminal state, as generated by the system equations, must produce a value for the optimal return function equal to or greater than the optimal value. Thus the control that minimizes the cost is the control which at each point of the trajectory causes the derivative of the optimal return function to be zero. This is the fundamental notion represented by the Hamilton-Jacobi-Bellman equation. Note that x and t can be either the initial or the current state and time, respectively. In this context, it will be used to represent the current state and time. Also note that every admissible trajectory must satisfy the terminal constraints $\Psi(y_f, \tau_f) = 0$.

$P(x, t)$ can be expanded as a series expansion in ϵ as

$$P(x, t) = \sum_{i=0}^{\infty} P_i(x, t)\epsilon^i \qquad (2.6)$$

and the optimal control can also be expanded in a series expansion as

$$u^{opt}(x, P_x, t) = \sum_{i=0}^{\infty} u_i(x, t)\epsilon^i \qquad (2.7)$$

where u^{opt} is obtained by substituting Eq. (2.6) into Eq. (2.7) and expanding the function. Therefore, it is possible to obtain the control law in feedback form.

The zeroth-order control, u_o, is the optimal control for the zeroth-order problem where $\epsilon = 0$. If an analytic solution can be obtained for the zeroth-order problem then higher-order solutions for the control can be obtained by expanding the Hamilton-Jacobi-Bellman equation

$$P_t = \sum_{i=0}^{\infty} P_{i_t}(x, t)\epsilon^i = - \left(\sum_{i=0}^{\infty} P_{i_x}(x, t)\epsilon^i \right) \left(\sum_{i=0}^{\infty} f_i \epsilon^i + \sum_{i=1}^{\infty} g_i \epsilon^i \right) \qquad (2.8)$$

where the dynamics have been expressed as expansions of the form

$$f^{opt}(x, u^{opt}, t) = \sum_{i=0}^{\infty} f_i(x, u, t)\epsilon^i \qquad (2.9)$$

$$g^{opt}(x, u^{opt}, t) = \sum_{i=0}^{\infty} g_i(x, u, t)\epsilon^i \qquad (2.10)$$

Expanding Eq. (2.8) and collecting terms of equal powers in ϵ, produces the following set of linear, first-order, partial differential equations

$$\begin{aligned} P_{i_t} + P_{i_x} f_0^{opt} &= -\sum_{j=0}^{i-1} P_{j_x}(f_{i-j} + g_{i-j-1}) \quad i = 1, 2, \ldots \qquad (2.11) \\ &= R_i(x, t, P_{i-1}, \ldots, P_0) \end{aligned}$$

The expansion of the Hamilton-Jacobi-Bellman equation will be detailed in the next section.

2.1   Expansion of the H-J-B Equation

The solution to the optimal control problem requires the evaluation of the Lagrange multiplier, P_x. Note that the quantity P_x is the partial derivative of the optimal return function with respect to the state y at the initial time or the current time (since at $\tau = t$, $y = x$). The function P_x is expanded in a series in the small parameter ϵ. The terms of this series expansion, P_{i_x}, are evaluated in terms of quadrature integrals which are functions of R_i. Recall that the functions R_i require the previously evaluated terms P_{j_x}, f_{i-j}, and g_{i-j-1} for $j = 1, \ldots, i - 1$. The coefficients f_i and g_i are the ith term in the series expansion of f and g given in Eqs. (2.9)-(2.10). Since f and g are assumed to be sufficiently differentiable, they are expressible in a power series in ϵ in terms

of the control. For a scalar control, this yields

$$f^{opt}(x, u^{opt}, t) = \sum_{i=0}^{\infty} \left[\left(\frac{1}{i!} \frac{\partial^i f(y, u, \tau)}{\partial u^i} \Big|_{x,t,\epsilon=0} \right) \left(\sum_{j=1}^{\infty} u_j \epsilon^j \right)^i \right] \qquad (2.12)$$

$$g^{opt}(x, u^{opt}, t) = \sum_{i=0}^{\infty} \left[\left(\frac{1}{i!} \frac{\partial^i g(y, u, \tau)}{\partial u^i} \Big|_{x,t,\epsilon=0} \right) \left(\sum_{j=1}^{\infty} u_j \epsilon^j \right)^i \right] \qquad (2.13)$$

The above equations assume that the zeroth-order control, u_0, is the dominant term in the series (Eq. (2.7)). This implies that the higher-order correction terms, u_1, u_2, $\cdots$, have a much smaller effect on the optimal return function, $P(x, t)$, than the zeroth-order term. The first four terms of f and g are obtained by use of Eqs. (2.12) and (2.13).

$$f_0 = f^{opt}(x, u_0, t) = f(x, u_0, t) \qquad (2.14)$$

$$f_1 = u_1 f_u(x, u_0, t) \qquad (2.15)$$

$$f_2 = \frac{u_1^2}{2} f_{uu}(x, u_0, t) + u_2 f_u(x, u_0, t) \qquad (2.16)$$

$$f_3 = \frac{u_1^3}{6} f_{uuu}(x, u_0, t) + u_1 u_2 f_{uu}(x, u_0, t)$$
$$+ u_3 f_u(x, u_0, t) \qquad (2.17)$$

$$\vdots$$

$$g_0 = g^{opt}(x, u_0, t) = g(x, u_0, t) \qquad (2.18)$$

$$g_1 = u_1 g_u(x, u_0, t) \qquad (2.19)$$

$$g_2 = \frac{u_1^2}{2} g_{uu}(x, u_0, t) + u_2 g_u(x, u_0, t) \qquad (2.20)$$

$$g_3 = \frac{u_1^3}{6} g_{uuu}(x, u_0, t) + u_1 u_2 g_{uu}(x, u_0, t)$$
$$+ u_3 g_u(x, u_0, t) \qquad (2.21)$$

$$\vdots$$

Note that in taking the partials with respect to u in Eqs. (2.12) and (2.13), the partial is taken first and then the partial is evaluated at x, t with ϵ set equal to zero. In other words, the partials are evaluated along the zeroth-order path.

2.2 Solution by the Method of Characteristics

The H-J-B equation (Eq. (2.5)) is a first-order partial differential equation. The expansion of the H-J-B equation results in the first-order differential equation for P_i stated in Eq. (2.11) with the boundary condition $P_i(x_f, t_f) = 0$, for $i = 1, \cdots$. Recall that f_0^{opt} denotes the dynamics of the zeroth-order problem ($\epsilon = 0$) using the zeroth-order control $u = u_0$. Recall also that the forcing term R_i is only a function of expansion terms of P of order less than i.

The method of characteristics is used to solve a set of linear or quasi-linear partial differential equations. This technique [11] requires the identification and solution of characteristics curves. The characteristic direction ds is defined by the equation

$$P_{i_t}(d\tau)_s + P_{i_x}(dy)_s = (dP_i)_s, \qquad i = 1, 2, \cdots \qquad (2.22)$$

Eqs. (2.11) along with (2.22) can be put in the form

$$\begin{bmatrix} 1 & f_0 \\ (d\tau)_s & (dy)_s \end{bmatrix} \begin{bmatrix} P_{i_t} \\ P_{i_x} \end{bmatrix} = \begin{bmatrix} R_i \\ (dP_i)_s \end{bmatrix} \qquad (2.23)$$

The characteristic directions for Eq. (2.23) are given by the solution of the differential equation that is obtained by setting the determinant of the matrix given in Eq. (2.23) equal to zero, such that

$$(dy)_s - f_0(d\tau)_s = 0 \quad \implies \quad (dy/d\tau)_s = f_0 \qquad (2.24)$$

The subscript s denotes the characteristic direction. Therefore, the characteristic curves of the equations, for any order term of P_i, are given by the zeroth-order optimal trajectory

$$\dot{y}_0 = f_0 \tag{2.25}$$

whose solution is denoted as $y_0(\tau; x, t)$.

The solution for P_i is given by

$$P_i(x, t) = -\int_t^{t_f} R_i^0 d\tau \tag{2.26}$$

where R_i^0 is defined along the zeroth-order path as

$$R_i^0 = R_i(y_0, \tau, P_{i-1}(y_0, \tau), \cdots, P_0(y_0, \tau)), \qquad i = 1, 2, \cdots \tag{2.27}$$

Therefore, having already determined P terms of order less than i, a solution for P_i can be determined by integrating R_i from the current 'time' to the final 'time' along the zeroth-order path.

2.3   Determination of the Optimal Control

Since the primary and secondary dynamics, f and g, are expanded in terms of the control (Eqs. (2.12) and (2.13)), the control expansion terms $u_0, u_1, u_2, \cdots$, need to be determined. The optimality condition provides the necessary tool to obtain these control terms. It can be stated as

$$P_x[f_u + \epsilon g_u] = \left[\sum_{i=0}^{\infty} P_{i_x} \epsilon^i\right]\left[\sum_{i=0}^{\infty}(f_{i_u} + \epsilon g_{i_u})\epsilon^i\right] = 0 \tag{2.28}$$

By expanding and multiplying out the terms of the two power series and equating like powers of ϵ, the following relations are obtained

$$\epsilon^0 \qquad : \qquad P_{0_x} f_u = 0 \tag{2.29}$$

$$\epsilon^1 \quad : \quad P_{0_x}[g_u + u_1 f_{uu}] + P_{1_x} f_u = 0 \tag{2.30}$$

$$\epsilon^2 \quad : \quad P_{0_x}\left[u_1 g_{uu} + \frac{1}{2}u_1^2 f_{uuu} + u_2 f_{uu}\right]$$

$$+ P_{1_x}[g_u + u_1 f_{uu}] + P_{2_x} f_u = 0 \tag{2.31}$$

$$\vdots \qquad\qquad\qquad \vdots$$

Note that u_0, the optimal control for the zeroth-order problem, can be solved using Eq. (2.29). Similarly, u_1 can be solved using Eq. (2.30) and u_2 can be solved using Eq. (2.31).

2.4 Determination of the Forcing Functions

Eqs. (2.14)–(2.21) and (2.29)–(2.31) can be used to solve for the forcing functions R_i where Eq. (2.11) can be restated as

$$R_i = -\sum_{j=0}^{i-1} P_{j_x}(f_{i-j} + g_{i-j-1}) \qquad i = 1, 2, \ldots \tag{2.32}$$

Using the above equations, R_1 is

$$R_1 = -P_{0_x}(f_1 + g_0) = -P_{0_x}(u_1 f_u + g) \tag{2.33}$$

With the use of the optimality condition of Eq. (2.29), R_1 becomes

$$R_1 = -P_{0_x} g_0 \tag{2.34}$$

Similarly, the equation for R_2 is

$$R_2 = -P_{0_x}(f_2 + g_1) - P_{1_x}(f_1 + g_0) \tag{2.35}$$

R_2 simplifies to the following equation when Eqs. (2.14)–(2.21) and (2.29)–(2.30) are substituted into the previous equation.

$$R_2 = \frac{u_1^2}{2} P_{0_x} f_{uu} - P_{1_x} g_0 \tag{2.36}$$

Finally, R_3 can be expressed as

$$R_3 = -P_{0_x}(f_3 + g_2) - P_{1_x}(f_2 + g_1) - P_{2_x}(f_1 + g_0) \tag{2.37}$$

This simplifies to

$$
\begin{aligned}
R_3 &= P_{0_x}\left[\frac{u_1^3}{3}f_{uuu} + u_1 u_2 f_{uu} - \frac{u_1^2}{2}g_{uu}\right] + P_{1_x}\left[\frac{u_1^2}{2}f_{uu}\right] - P_{2_x}g_0 \\
&= P_{0_x}\left[\frac{u_1^3}{12}f_{uuu} + \frac{u_1 u_2}{2}f_{uu}\right] - P_{1_x}\left[\frac{u_1}{2}g_u\right] - P_{2_x}\left[g_0 + \frac{u_1}{2}f_u\right] \tag{2.38}
\end{aligned}
$$

Using the expression for R_i, the expression for the Lagrange multipliers, P_{i_x}, can be expressed as

$$P_{i_x} = \frac{\partial P_i}{\partial x} = -\int_t^{t_f} \frac{\partial R_i}{\partial x}d\tau + R_i|_t \frac{\partial t}{\partial x} - R_i|_{t_f}\frac{\partial t_f}{\partial x} \tag{2.39}$$

Once these P_{i_x} are determined, they can be used in the optimal control expansion (Eq. (2.7)). As made apparent in the above equations, the solution becomes increasingly complex as the higher-order correction terms rely on the state information from the lower-order trajectories.

Chapter 3

Modelling of the ALS Configuration

This chapter presents the modelling characteristics and the equations of motion for the rocket. Included are sections on the properties of the propulsion, aerodynamics, masses, gravity, and the atmosphere. A small expansion parameter, the ratio of the atmospheric scale height to the radius of the Earth, is then used to separate the dynamics into the primary and perturbation effects. Lastly, the equations of motion for the zeroth-order problem of flight in a vacuum over a flat Earth are presented.

The Advanced Launch System (ALS) is designed to be an all-weather, unmanned, two-stage launch vehicle for placing medium payloads into a low Earth orbit. The spacecraft (fig. 3.1) consists of a liquid rocket booster with seven engines and a core vehicle that contains three engines. All ten liquid hydrogen/liquid oxygen low cost engines are ignited at launch. Staging occurs when the booster's seven engines have exhausted their propellant. The three core engines burn continuously from launch until they are shut down at orbital insertion. Launched in the equatorial plane and ending at the perigee of a 80nm by 150nm transfer orbit, the flight occurs in two-dimensions over a nonrotating, spherical Earth. Note, the booster is assumed to ride on top of the core throughout the first stage trajectory.
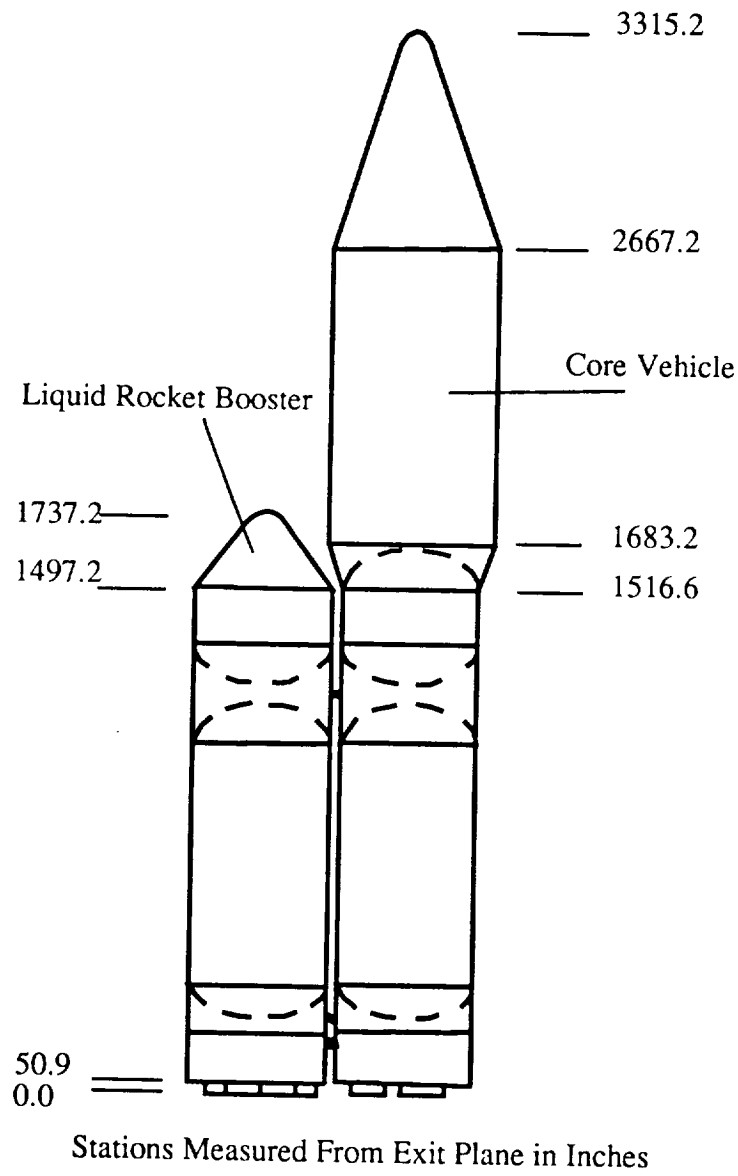
14

3315.2

2667.2

Core Vehicle

Liquid Rocket Booster

1737.2

1683.2

1497.2

1516.6

50.9

0.0

Stations Measured From Exit Plane in Inches

Figure 3.1: ALS Vehicle Configuration

3.1 Equations of Motion for the Launch Problem

The general equations of motion for a launch vehicle modelled as a point mass over a spherical, nonrotating Earth are given for flight in three-dimensions as

$$\dot{h} = V \sin \gamma \tag{3.1}$$

$$\dot{V} = \frac{(T \cos \alpha \cos \beta - D)}{m} - g \sin \gamma \tag{3.2}$$

$$\dot{\gamma} = \frac{[-(T \cos \alpha \sin \beta - Q) \sin \mu + (T \sin \alpha + L) \cos \mu]}{mV}$$
$$+ [\frac{V}{(r_e + h)} - \frac{g}{V}] \cos \gamma \tag{3.3}$$

$$\dot{\chi} = \frac{[(T \cos \alpha \sin \beta - Q) \cos \mu + (T \sin \alpha + L) \sin \mu]}{(mV \cos \gamma)}$$
$$+ \frac{V \tan \phi \cos \gamma \cos \chi}{(r_e + h)} \tag{3.4}$$

$$\dot{\theta} = \frac{V \cos \gamma \cos \chi}{(r_e + h) \cos \phi} \tag{3.5}$$

$$\dot{\phi} = -\frac{V \cos \gamma \sin \chi}{(r_e + h)} \tag{3.6}$$

$$\dot{m} = -\sigma T_{vac} \tag{3.7}$$

The vehicle coordinate system is shown in figure 3.2. Note, the engines are not gimbaled and the aerodynamic pitching moments are neglected. For a vertical launch Eqs. (3.3)-(3.4) experience a singularity caused by the velocity being zero and by a flight path angle of 90 degrees, respectively. Therefore, a pitch-over maneuver must be made at launch and equations of motion written in a different coordinate frame must be used.
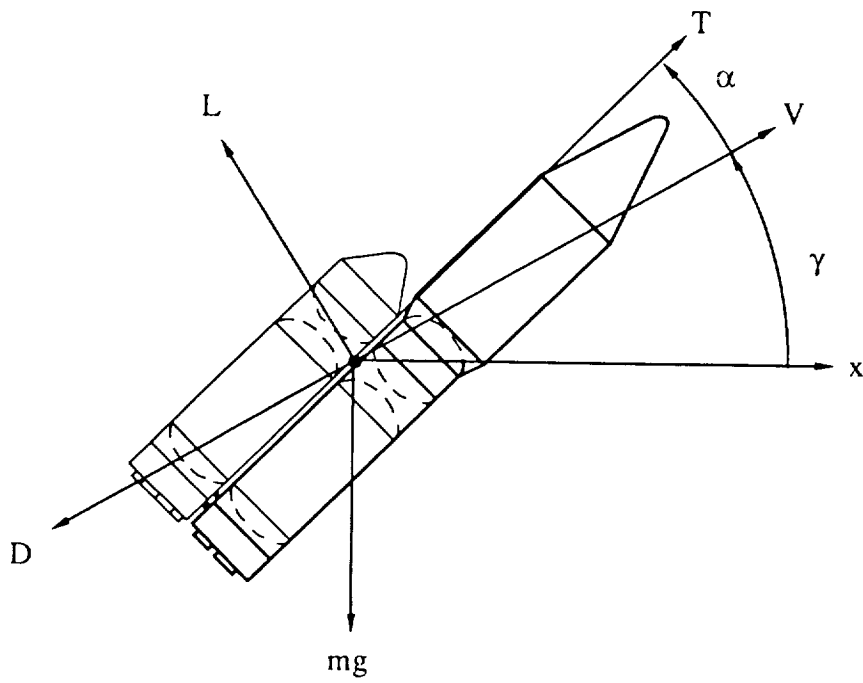
Figure 3.2: Coordinate Axis Definition

3.2   Propulsion

Thrust is assumed to act along the centerline of the booster-core vehicle configuration and to be the same constant value for each engine. The total thrust of the rocket changes after staging as the seven engines of the booster are discarded, leaving only the three engines of the core vehicle.

$$T = (T_{vac} - npA_e) \qquad T_{vac} = n \times 580,110. \text{ lbs.}$$

where T_{vac} is the total value of the thrust when acting in a vacuum and the number of engines is $n = 10$ for the first stage and $n = 3$ for the second stage. Notice the variation of the thrust due to the atmospheric pressure p is given for an underexpanded nozzle and thus a conservative value for thrust is used. The value of the engine nozzle exit area is $A_e = 5814.8/144.$ sq ft. The specific fuel consumption of the rocket is

$$\sigma = \frac{1}{I_{sp}\, g_s} \, \frac{\text{sec}}{\text{ft}} \tag{3.8}$$

and the specific impulse $I_{sp} = 430.$ seconds. The value of σ remains the same after staging occurs.

3.3   Aerodynamics

Since sideslip causes drag, the vehicle is assumed to fly at zero sideslip angle, so that only the angle-of-attack gives the orientation of the vehicle relative to the free stream. The direction of the lift vector is then controlled through the velocity roll angle. With no sideslip, the side force Q is identically zero. Therefore,
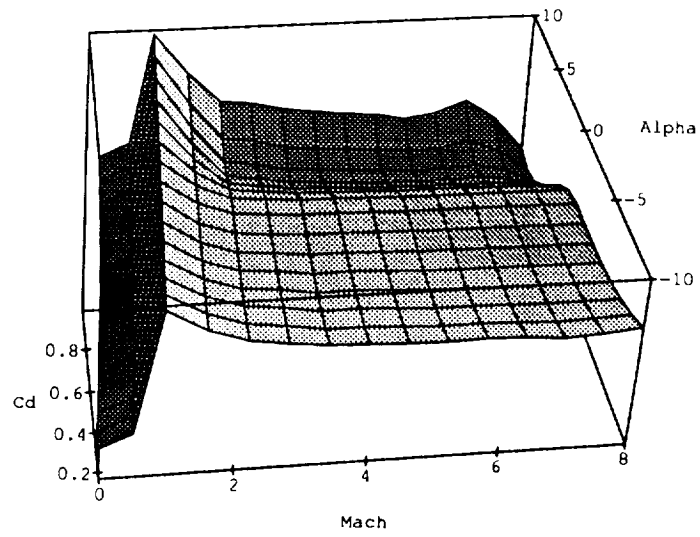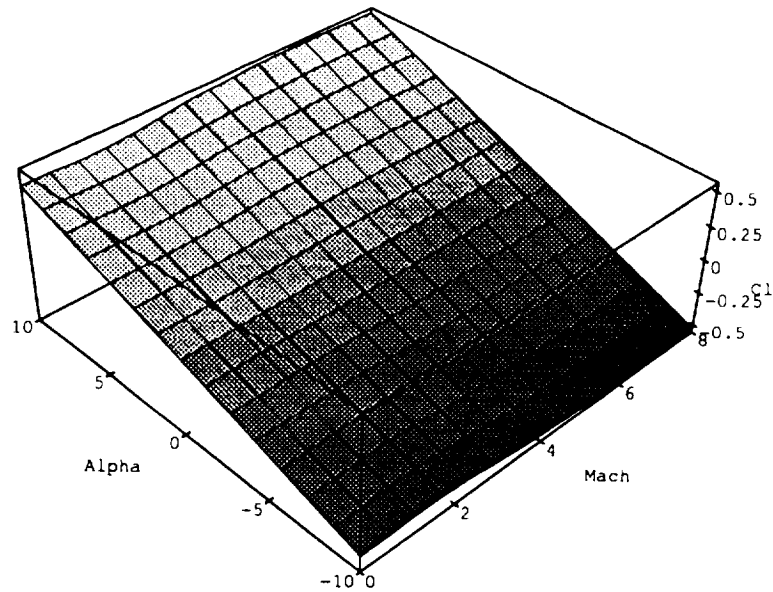
Figure 3.3: First Stage Drag Model

Figure 3.4: First Stage Lift Model

$$L = C_L qS, \quad D = C_D qS, \quad Q = C_Q qS = 0 \qquad (3.9)$$

where C_L, C_D, C_Q are the lift, drag, and side force coefficients, respectively, S is the cross-sectional area of the combined vehicle (booster + core), and $q = \frac{1}{2}\rho V^2$ is the dynamic pressure. The cross-sectional area S is assumed to be the same constant value before and after staging occurs.

The aerodynamic data has been provided in tabular form [4] and is modelled by polynomials in α with Mach-number-dependent coefficients. For the first stage, the aerodynamic coefficients are written as

$$
\begin{aligned}
C_D(M,\alpha) &= C_{D_o}(M) + C_{D_{\alpha 2}}(M)\alpha^2 + C_{D_{\alpha 3}}(M)\alpha^3 \\
C_L(M,\alpha) &= C_{L_\alpha}(M)\alpha
\end{aligned}
\qquad (3.10)
$$

where the Mach-number-dependent terms have been obtained from cubic-spline curve fits of the tabular data. Three-dimensional plots [12] of the first stage drag and lift models are shown in Figures 3.3 and 3.4. Note that the drag coefficient of this vehicle at supersonic and hypersonic speeds has a minimum at a positive angle of attack as shown in Figure 3.3. This is caused by the aerodynamic shielding of the booster by the flow field of the core.

After staging, the vehicle operates in the hypersonic flow regime and the aerodynamic force coefficients are modelled as

$$
\begin{aligned}
C_D(\alpha) &= C_{D_o} + C_{D_\alpha}\alpha + C_{D_{\alpha 2}}\alpha^2 \\
C_L(\alpha) &= C_{L_\alpha}\alpha + C_{L_{\alpha 2}}\alpha^2
\end{aligned}
\qquad (3.11)
$$

with constant coefficients $C_{D_o} = .2011$, $C_{D\alpha} = 0.0$, $C_{D_{\alpha 2}} = .001811$, $C_{L_\alpha} =$
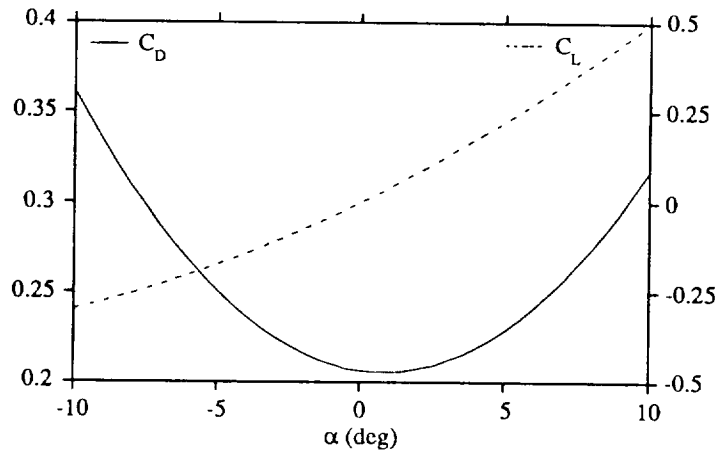
Figure 3.5: Second Stage Aerodynamic Model

.039962, and $C_{L_{\alpha^2}}$ = .00100272. The aerodynamic plot of C_L and C_D is provided in figure 3.5.

3.4 Mass Characteristics

The inert weights of the booster and core, the weight of the propellant, the payload and payload margin, and the weight of the payload fairing comprise the ALS takeoff weight. The fairing encases the payload and is carried along by the core vehicle until orbital insertion. The vehicle mass and sea-level weight characteristics are shown in Table 3.1. The time at which staging is to occur is obtained from the first stage mass flow rate and the propellant of the booster

$$t_{stage} = \frac{m_{propellant}}{7\sigma \tilde{T}_{vac}} = 153.54 \text{ sec.}$$

Vehicle Stage	Vehicle Component	Take-off Weight (lbs.)
Core	Inert Mass	176,130.00
	Propellant	1,479,180.00
	Payload	120,000.00
	Payload Margin	12,000.00
	Payload Faring	39,120.00
	Total Core	1,826,430.00
Booster	Inert Mass	216,880.00
	Propellant	1,449,980.00
	Total Booster	1,666,860.00
Core + Booster	Total at Take-off	3,493,290.00

Table 3.1: Vehicle Mass Characteristics

where the vacuum thrust per engine is $\tilde{T}_{vac} = 580110$.

Once the stage time, the total first stage mass flow rate, the takeoff weight, and the inert weight of the booster are known, then the weight of the vehicle at the end of the first stage and the initial weight in the second stage can be calculated. For this vehicle the values are

$$m_{stage1} = 1421890. \text{ lbs.}, \quad m_{stage2} = 1250010. \text{ lbs.}, \quad \Delta m_{stage} = 216880. \text{ lbs.}$$

3.5 Gravitational and Atmospheric Models

The gravitational acceleration is modelled as an altitude-varying function by the inverse square law,

$$g = g_s \frac{r_e^2}{(r_e + h)^2}$$

but will be assumed constant in the zeroth-order problem to facilitate obtaining an analytic solution. The constant values for gravity at sea-level and for the radius of the Earth are

$$g_s = 32.174 \, \frac{\text{ft}}{\text{sec}^2} \qquad r_e = 2.09256725 \times 10^7 \text{ ft.}$$

The atmospheric density is expressed by the exponential function,

$$\rho = \rho_r e^{-(r_e+h)/h_s} = \rho_r e^{-r_e/h_s} e^{-h/h_s} = \rho_s e^{-h/h_s} \qquad (3.12)$$

where h_s is the atmospheric scale height and ρ_s is the sea-level reference density. The values for these parameters are

$$\rho_s = .002377 \, \frac{\text{slugs}}{\text{ft}^3} \quad h_s = 23,800. \text{ ft.}$$

The form of the density is chosen to motivate the selection of a small parameter to exclude the aerodynamics in the zeroth-order dynamics. If ϵ is chosen as

$$\epsilon = h_s/r_e \qquad (3.13)$$

and defining

$$\delta(\epsilon, h) = \frac{\rho(\epsilon, h)}{\epsilon} \qquad (3.14)$$

then by atmospheric properties $\delta(\epsilon, h) > 0$. The exponential density also satisfies the requirement [3] that the perturbation term in the dynamics remains small, i.e.,

$$\lim_{\epsilon \to 0} \delta(\epsilon, h) \to 0 \qquad (3.15)$$

Satisfaction of this property will allow more general atmospheric models to be used in the launch problem.

The atmospheric pressure is also expressed as an exponential function,

$$p = p_s e^{-h/h_p} \tag{3.16}$$

where h_p is the atmospheric pressure scale height and p_s is the sea-level reference pressure. The values for these parameters are

$$p_s = 2116.24 \frac{\text{lbs}}{\text{ft}^2} \quad h_p = 23,200. \text{ ft}.$$

The speed of sound can be obtained by the relationship

$$sos = \sqrt{\frac{\Gamma p}{\rho}}$$

with the specific heat ratio for air given as $\Gamma = 1.4$.

The gravity can be rewritten as

$$g = g_s - \frac{g_s h(2r_e + h)}{(r_e + h)^2} = g_s - \frac{\epsilon g_s h(2r_e + h)r_e}{h_s(r_e + h)^2} \tag{3.17}$$

where the expansion parameter has formally been introduced and the second term is clearly small in comparison to the first term which is the value for gravity at sea-level, g_s.

3.6 Expansion Dynamics

In terms of the small parameter ϵ, the full-order equations of motion are rewritten as

$$\dot{h} = V \sin\gamma \tag{3.18}$$

$$\dot{V} = \frac{T_{vac}}{m} \cos\alpha \cos\beta - g_s \sin\gamma \tag{3.19}$$
$$+ \epsilon \left[-\frac{npA_e r_e}{mh_s} \cos\alpha \cos\beta + \frac{g_s h(2r_e + h)r_e \sin\gamma}{h_s(r_e + h)^2} - \frac{\rho S V^2 C_D r_e}{2mh_s} \right]$$

$$\dot{\gamma} = -\frac{T_{vac}}{mV}(\cos\alpha\sin\beta\sin\mu - \sin\alpha\cos\mu) - \frac{g_s\cos\gamma}{V}$$

$$-\epsilon\frac{npA_e r_e}{mVh_s}(\cos\alpha\sin\beta\sin\mu - \sin\alpha\cos\mu)$$

$$+\epsilon\frac{\rho SVr_e}{2mh_s}(C_Q\sin\mu + C_L\cos\mu)$$

$$+\epsilon\left(\frac{V}{r_e + h} + g_s\frac{h(2r_e + h)}{V(r_e + h)^2}\right)\frac{r_e}{h_s}\cos\gamma \qquad (3.20)$$

$$\dot{\chi} = \frac{T_{vac}}{mV\cos\gamma}(\cos\alpha\sin\beta\cos\mu + \sin\alpha\sin\mu)$$

$$-\epsilon\frac{npA_e r_e}{mVh_s\cos\gamma}(\cos\alpha\sin\beta\cos\mu + \sin\alpha\sin\mu)$$

$$+\epsilon\left[\frac{\rho SVr_e}{mh_s\cos\gamma}(C_L\sin\mu - C_Q\cos\mu) + \frac{Vr_e\tan\phi\cos\gamma\cos\chi}{h_s(r_e + h)}\right] \qquad (3.21)$$

$$\dot{\theta} = \frac{V\cos\gamma\cos\chi}{r_e\cos\phi}(1 - \epsilon\frac{h}{h_s}) \qquad (3.22)$$

$$\dot{\phi} = -\frac{V\cos\gamma\sin\chi}{r_e}(1 - \epsilon\frac{h}{h_s}) \qquad (3.23)$$

Where the binomial formula has been used to rewrite $(r_e + h)^{-1}$ for the longitude and latitude since $r_e \gg h$.

3.6.1 Two-Dimensional Flight

In this section the three-dimensional equations of motion are reduced for flight in a great-circle plane (the X-Z plane) over a flat, nonrotating Earth. If the vehicle is assumed to be restricted to fly in the equatorial plane then the lift, thrust, and velocity vectors all lie in the same plane and the roll angle ($\mu = 0$) is eliminated from the equations. Under the previously mentioned assumptions of no side force ($Q = 0$) and no sideslip ($\beta = 0$), the zeroth-order equations of motion representing flight in a vacuum over a flat Earth become

$$\dot{h} = V\sin\gamma \qquad (3.24)$$

$$\dot{V} \;=\; \frac{T_{vac}}{m}\cos\alpha - g_s \sin\gamma \tag{3.25}$$

$$\dot{\gamma} \;=\; \frac{T_{vac}}{mV}\sin\alpha - \frac{g_s}{V}\cos\gamma \tag{3.26}$$

$$\dot{\theta} \;=\; \frac{V\cos\gamma}{r_e} \tag{3.27}$$

$$\dot{m} \;=\; -\sigma T_{vac} \Longrightarrow m = m_0 - \sigma T_{vac}(\tau - \tau_0) \tag{3.28}$$

$$\chi \;=\; \chi_0 = 0.0$$

$$\phi \;=\; \phi_0 = 0.0$$

These are the system dynamics used to obtain an analytic solution to the zeroth-order optimization problem presented in the next chapter.

Chapter 4

Zeroth-Order Optimization Problem

The solution to the zeroth-order optimization problem is derived by a coordinate transformation. A canonical transformation from the wind axis to the rectangular or local horizon coordinate frame allows the zeroth-order problem to be solved analytically. The solution is in closed form up to some constants that can be determined numerically to solve the two-point boundary value problem. The conditions for connecting the second stage subarc to the first stage subarc are then presented.

4.1 Optimization Problem Statement

In this section the zeroth-order optimization problem is presented. The problem is to maximize the payload into orbit

$$J = -m_f$$

subject to terminal constraints on the altitude, velocity, and flight path angle,

$$h_f = h_{f_{spec}}, \quad V_f = V_{f_{spec}}, \quad \gamma_f = \gamma_{f_{spec}}$$

subject to the state discontinuity in the mass at a interior point where staging occurs,

$$m_{stage2} = m_{stage1} - \Delta m_{stage}$$

and subject to the equations of motion for flight in the equatorial plane.

$$\dot{h} = V \sin \gamma \tag{4.1}$$

$$\dot{V} = \frac{T}{m} \cos \alpha - g_s \sin \gamma \tag{4.2}$$

$$\dot{\gamma} = \frac{T}{mV} \sin \alpha - \frac{g_s}{V} \cos \gamma \tag{4.3}$$

$$\dot{\theta} = \frac{V \cos \gamma}{r_e} \tag{4.4}$$

$$\dot{m} = -\sigma T \implies m = m_0 - \sigma T(\tau - \tau_0) \tag{4.5}$$

Note, in this section and when discussing the zeroth-order trajectory, the total vacuum thrust will be represented by T and the subscript notation will be dropped.

The Hamiltonian for this system can then be expressed as

$$H = \lambda_h V \sin \gamma + \lambda_V (\frac{T}{m} \cos \alpha - g_s \sin \gamma) + \lambda_\gamma (\frac{T}{mV} \sin \alpha - \frac{g_s}{V} \cos \gamma) \tag{4.6}$$

The zeroth-order control law determined by the optimality conditon is

$$H_\alpha = -\frac{T}{m} \lambda_V \sin \alpha + \frac{T}{mV} \lambda_\gamma \cos \alpha = 0 \tag{4.7}$$

By the strengthened Legendre-Clebesch condition $H_{\alpha\alpha} > 0$ choose

$$\tan \alpha = \frac{\lambda_\gamma}{V \lambda_V}$$

$$\cos \alpha = -\frac{V \lambda_V}{\sqrt{(V \lambda_V)^2 + \lambda_\gamma^2}}$$

$$\sin \alpha = -\frac{\lambda_\gamma}{\sqrt{(V \lambda_V)^2 + \lambda_\gamma^2}} \tag{4.8}$$

Whereas the optimal control can be derived in terms of the states and Lagrange multipliers, an analytic solution is not possible for the states and Lagrange

multipliers written in the wind axis frame. Therefore, a coordinate transformation into the Cartesian reference frame is presented in the next section. In section 4.3 an analytic solution is obtained using this transformation.

4.2 Zeroth-Order Coordinate Transformation

The analytic solution for the zeroth-order problem can be found in the Cartesian coordinate system but the equations of motion of the full system which include the aerodynamic forces are written in the wind axis system. Therefore, to derive the zeroth-order control and the first-order correction to the control the transformation of coordinates and especially the transformation of the Lagrange multipliers must be known. This can be accomplished by a canonical transformation [see appendix B] from the (θ, ϕ, h) coordinates to the right-handed coordinate system (X, Y, Z), where X is positive in an eastward direction along the equator, Z is positive pointing towards the Earth, and Y is orthogonal to the $X - Z$ plane. The relationship between the two reference frames (see figure 4.2) is $X = r_e\theta$, $Y = r_e\phi$, and $Z = -h$. In two-dimensions, the corresponding velocity coordinates (u,w) are considered positive in the positive X and Z directions, respectively. A necessary and sufficient condition [13] for a canonical transformation is the equivalence of the Hamiltonians in the two reference frames.

$$H_{LH} = \lambda_X dX + \lambda_Y dY + \lambda_h dh + \lambda_u du + \lambda_w dw \qquad (4.9)$$

$$H_{Wind} = \lambda_\theta d\theta + \lambda_\phi d\phi + \lambda_h dh + \lambda_V dV + \lambda_\gamma d\gamma \qquad (4.10)$$
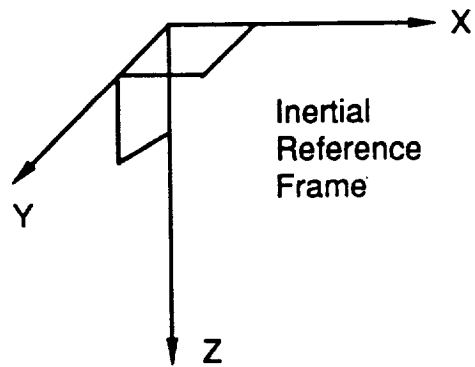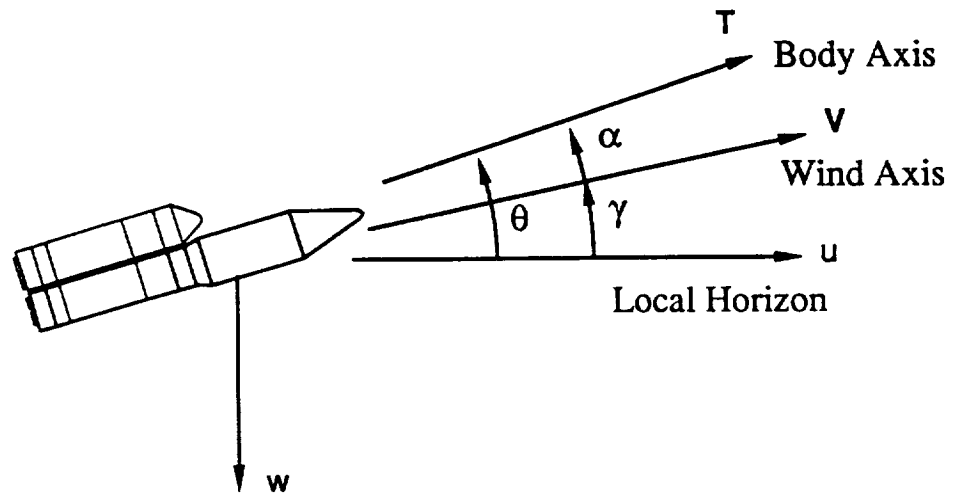
Figure 4.1: Transformation of Coordinate Systems

This equivalence is obtained through the Jacobian of the transformation. Therefore, the transformation

$$u = V \cos \gamma, \qquad w = -V \sin \gamma \tag{4.11}$$

requires

$$\begin{bmatrix} \lambda_V \\ \lambda_\gamma \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial V} & \frac{\partial w}{\partial V} \\ \frac{\partial u}{\partial \gamma} & \frac{\partial w}{\partial \gamma} \end{bmatrix} \begin{bmatrix} \lambda_u \\ \lambda_w \end{bmatrix}$$

and thus,

$$\begin{bmatrix} \lambda_V \\ \lambda_\gamma \end{bmatrix} = \begin{bmatrix} \cos \gamma & -\sin \gamma \\ -V \sin \gamma & -V \cos \gamma \end{bmatrix} \begin{bmatrix} \lambda_u \\ \lambda_w \end{bmatrix}$$

This produces the transformation of the Lagrange multipliers,

$$\lambda_V = \lambda_u \cos \gamma - \lambda_w \sin \gamma \tag{4.12}$$

$$\lambda_\gamma = -V(\lambda_u \sin \gamma + \lambda_w \cos \gamma) \tag{4.13}$$

$$\lambda_\theta = r_e \lambda_X \tag{4.14}$$

$$\lambda_\phi = r_e \lambda_Y \tag{4.15}$$

and the transformation of the states,

$$V = \sqrt{u^2 + w^2} \tag{4.16}$$

$$\sin \gamma = -\frac{w}{V} \tag{4.17}$$

4.3 Zeroth-Order Analytic Solution in the Cartesian Frame

In this section an analytic solution will be derived for the zeroth-order problem of maximum payload into orbit for flight in a vacuum over a flat Earth. This solution is made possible by the coordinate transformation presented in

the previous section. The equations of motion in a Cartesian coordinate frame are

$$\dot{X} = u \tag{4.18}$$

$$\dot{Y} = 0 \Longrightarrow Y = Y_0 = 0$$

$$\dot{h} = -w \tag{4.19}$$

$$\dot{u} = \frac{T}{m}\cos\theta_p \tag{4.20}$$

$$\dot{v} = 0 \Longrightarrow v = v_0 = 0$$

$$\dot{w} = -\frac{T}{m}\sin\theta_p + g_s \tag{4.21}$$

$$\dot{m} = -\sigma T \Longrightarrow m = m_0 - \sigma T(\tau - \tau_0) \tag{4.22}$$

The Hamiltonian is

$$H = \lambda_X u - \lambda_h w + \lambda_u \frac{T}{m}\cos\theta_p + \lambda_w(-\frac{T}{m}\sin\theta_p + g_s) \tag{4.23}$$

The zeroth-order control law is determined by the optimality conditon

$$H_{\theta_p} = -\frac{T}{m}\lambda_u \sin\theta_p - \frac{T}{m}\lambda_w \cos\theta_p = 0 \tag{4.24}$$

Therefore, using the strengthened Legendre-Clebesch condition the control becomes

$$\tan\theta_p = -\frac{\lambda_w}{\lambda_u}$$

$$\cos\theta_p = -\frac{\lambda_u}{\sqrt{\lambda_u^2 + \lambda_w^2}}$$

$$\sin\theta_p = \frac{\lambda_w}{\sqrt{\lambda_u^2 + \lambda_w^2}} \tag{4.25}$$

The Lagrange multipliers are obtained using $\dot{\lambda}_y = -H_y^T$

$$\dot{\lambda}_X = 0$$

$$\dot{\lambda}_h = 0$$

$$\dot{\lambda}_u = -\lambda_X$$

$$\dot{\lambda}_w = \lambda_h$$

with the boundary conditions

$$\lambda_X(\tau_f) = \nu_X, \quad \lambda_h(\tau_f) = \nu_h, \quad \lambda_u(\tau_f) = \nu_u, \quad \lambda_w(\tau_f) = \nu_w$$

where $\nu_X, \nu_h, \nu_u, \nu_w$ are unknown Lagrange multipliers associated with the terminal constraints. For the unconstrained downrange problem, the solutions to the adjoint differential equations are

$$\lambda_X = \nu_X = 0$$

$$\lambda_h = \nu_h \tag{4.26}$$

$$\lambda_u = \nu_u = C_u \tag{4.27}$$

$$\lambda_w = C_w + \lambda_h(\tau - \tau_0) \tag{4.28}$$

The equations of motion can be integrated by changing the independent variable from time to mass and using the mass equation (Eq. (4.5)) to substitute mass for τ. As a consequence, the Lagrange multipliers are rewritten as

$$\lambda_u = C_u \tag{4.29}$$

$$\lambda_w = \overline{C}_w - \lambda_h \frac{m}{\sigma T} \tag{4.30}$$

$$\lambda_u^2 + \lambda_w^2 = cm^2 + bm + a \tag{4.31}$$

where

$$c = \frac{\lambda_h^2}{(\sigma T)^2} \tag{4.32}$$

$$b = -\frac{2}{\sigma T}\lambda_h\overline{C}_w \tag{4.33}$$

$$a = C_u^2 + \overline{C}_w^2 \tag{4.34}$$

$$\overline{C}_w = C_w + \lambda_h\frac{m_0}{\sigma T} \tag{4.35}$$

The derivatives of the states with respect to mass are

$$\frac{du}{dm} = \frac{C_u}{\sigma m\sqrt{cm^2 + bm + a}} \tag{4.36}$$

$$\frac{dw}{dm} = \frac{\lambda_w}{\sigma m\sqrt{cm^2 + bm + a}} - \frac{g_s}{\sigma T} \tag{4.37}$$

$$\frac{dX}{dm} = -\frac{u}{\sigma T} \tag{4.38}$$

$$\frac{dh}{dm} = \frac{w}{\sigma T} \tag{4.39}$$

Note that $c > 0$, $a > 0$, and the discriminant of the quadratic mass equation $\Delta \triangleq 4ac - b^2 > 0$ since

$$\Delta = \frac{4}{(\sigma T)^2}\left(\lambda_h C_u\right)^2 \tag{4.40}$$

From these differential equations the solution is found from standard integrals.

$$u = u_0 - \frac{C_u}{\sigma\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a + bm}{m\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2a + bm_0}{m_0\sqrt{\Delta}}\right)\right] \tag{4.41}$$

$$w = w_0 - \frac{g_s}{\sigma T}(m - m_0)$$
$$\qquad - \frac{\lambda_h}{\sigma^2 T\sqrt{c}}\left[\sinh^{-1}\left(\frac{2cm + b}{\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2cm_0 + b}{\sqrt{\Delta}}\right)\right]$$
$$\qquad - \frac{\overline{C}_w}{\sigma\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a + bm}{m\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2a + bm_0}{m_0\sqrt{\Delta}}\right)\right] \tag{4.42}$$

$$h = h_0 - \frac{g_s}{2(\sigma T)^2}(m - m_0)^2 + \frac{(m - m_0)}{\sigma T}w_0$$

$$-\frac{\lambda_h}{\sigma(\sigma T)^2\sqrt{c}}\left(m+\frac{b}{2c}\right)\left[\sinh^{-1}\left(\frac{2cm+b}{\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2cm_0+b}{\sqrt{\Delta}}\right)\right]$$

$$-m\frac{\overline{C}_w}{\sigma(\sigma T)\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$

$$-\frac{\overline{C}_w}{\sigma(\sigma T)\sqrt{c}}\left[\sinh^{-1}\left(\frac{2cm+b}{\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2cm_0+b}{\sqrt{\Delta}}\right)\right]$$

$$-\frac{\lambda_h}{\sigma(\sigma T)^2 c}\left[\sqrt{cm_0^2+bm_0+a}-\sqrt{cm^2+bm+a}\right] \qquad (4.43)$$

$$X = X_0 - \frac{(m-m_0)}{\sigma T}u_0$$

$$+m\frac{C_u}{\sigma(\sigma T)\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$

$$+\frac{C_u}{\sigma(\sigma T)\sqrt{c}}\left[\sinh^{-1}\left(\frac{2cm+b}{\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2cm_0+b}{\sqrt{\Delta}}\right)\right]$$

The equation for the altitude can be manipulated further to eliminate some common terms.

$$h = h_0 - \frac{g_s}{2(\sigma T)^2}(m-m_0)^2 + \frac{(m-m_0)}{\sigma T}w_0$$

$$-m\frac{\lambda_h}{\sigma(\sigma T)^2\sqrt{c}}\left[\sinh^{-1}\left(\frac{2cm+b}{\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2cm_0+b}{\sqrt{\Delta}}\right)\right]$$

$$-m\frac{\overline{C}_w}{\sigma(\sigma T)\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$

$$-\frac{\lambda_h}{\sigma(\sigma T)^2 c}\left[\sqrt{cm_0^2+bm_0+a}-\sqrt{cm^2+bm+a}\right]$$

At the final time, $H_f = -1$ by the transversality condition. Using the Hamiltonian and the three state equations u,w, and h, which have prescribed initial and final values, the four unknown constants associated with the two-point boundary value problem can be solved. For the problem of flight restricted to a plane, the unknowns are m_f, C_u, C_w, and λ_h. The analytic state equations (Eq. (4.41)-(4.43)) are nonlinear and thus no statement can be made about the

existence or uniqueness of the set of constants found. Therefore, if multiple solutions are found the solution set which minimizes the Hamiltonian would be chosen. At the very least, the Legendre-Clebesch condition, $H_{uu} \geq 0$, for a weak relative minimum must be satisfied.

4.4 Linking the First and Second Stage Subarcs

Of interest in this section is the linking of the two subarcs of the two-stage rocket. By the corner conditions, the Lagrange multipliers for all the states must be continuous.

$$\lambda_y(t_{stage}-) = \lambda_y(t_{stage}+) \tag{4.44}$$

The analytic solution previously presented is still valid for either subarc but only by using this relationship between the Lagrange multipliers can the second stage be connected to the first stage subarc. Recall that the constant C_w is associated with the initial condition of the Lagrange multiplier for the vertical velocity component. For a subarc with first stage initial conditions, the equations become

$$\lambda_w(t_{stage}) = C_w + \lambda_h(t_{stage} - t_0) \tag{4.45}$$

$$\lambda_w(t) = \lambda_w(t_{stage}) + \lambda_h(t - t_{stage}) \quad t \geq t_{stage}+ \tag{4.46}$$

Rewriting the Lagrange multipliers using the corner condition and with mass replacing time as the independent variable, results in

$$\lambda_h = \nu_h = \text{constant} \qquad t_0 \leq t \leq t_f \tag{4.47}$$

$$\lambda_u = \nu_u = C_u = \text{constant} \qquad t_0 \leq t \leq t_f \tag{4.48}$$

$$\lambda_w = C_w + \frac{\lambda_h}{\sigma T_1}(m_0 - m) \qquad t_0 \leq t \leq t_{stage} \qquad (4.49)$$

$$\lambda_w(t+) = C_w + \frac{\lambda_h}{\sigma T_1}(m_0 - m_{stage1}) + \frac{\lambda_h}{\sigma T_2}(m_{stage2} - m) \qquad (4.50)$$

where T_1 and T_2 represent the thrust for the first and second stages, respectively. The equations of motion, written with mass as the independent variable, which were previously presented are still valid but the constant coefficients of the quadratic equation are of a different form.

$$\lambda_u^2 + \lambda_w^2 = c'm^2 + b'm + a' \qquad (4.51)$$

$$c' = \frac{\lambda_h^2}{(\sigma T_2)^2} \qquad (4.52)$$

$$b' = -\frac{2}{\sigma T_2}\lambda_h \overline{C}_w' \qquad (4.53)$$

$$a' = C_u^2 + \overline{C}_w'^2 \qquad (4.54)$$

$$\overline{C}_w' = C_w + \lambda_h \frac{m_0}{\sigma T_1} + \lambda_h(\frac{m_{stage2}}{\sigma T_2} - \frac{m_{stage1}}{\sigma T_1}) \qquad (4.55)$$

$$= \overline{C}_w + \lambda_h(\frac{m_{stage2}}{\sigma T_2} - \frac{m_{stage1}}{\sigma T_1})$$

Therefore, the state equations become

$$\frac{du}{dm} = \frac{C_u}{\sigma m \sqrt{c'm^2 + b'm + a'}}$$

$$\frac{dw}{dm} = \frac{\lambda_w}{\sigma m \sqrt{c'm^2 + b'm + a'}} - \frac{g_s}{\sigma T_2}$$

$$\frac{dX}{dm} = -\frac{u}{\sigma T_2}$$

$$\frac{dh}{dm} = \frac{w}{\sigma T_2}$$

The same standard integrals apply to the solution of the problem because $a' > 0$, $c' > 0$ and the discriminant

$$\Delta' = 4a'c' - b'^2 = 4\left(\frac{\lambda_h}{\sigma T_2}\right)^2 C_u^2 > 0 . \qquad (4.56)$$

The simplified form of the solution to the state equations (Eqs. (4.41)-(4.43)) is also still valid but with the first stage subarc used as the initial conditions of the second stage subarc.

$$
u = u_0 - \frac{C_u}{\sigma\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a + bm_{stage1}}{m_{stage1}\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2a + bm_0}{m_0\sqrt{\Delta}}\right)\right]
$$
$$
- \frac{C_u}{\sigma\sqrt{a'}}\left[\sinh^{-1}\left(\frac{2a' + b'm}{m\sqrt{\Delta'}}\right) - \sinh^{-1}\left(\frac{2a' + b'm_{stage2}}{m_{stage2}\sqrt{\Delta'}}\right)\right] \quad (4.57)
$$

$$
w = w_0 - \frac{g_s}{\sigma T_1}(m_{stage1} - m_0)
$$
$$
+ \frac{\overline{C}_w}{\sigma\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a + bm_{stage1}}{m_{stage1}\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2a + bm_0}{m_0\sqrt{\Delta}}\right)\right]
$$
$$
- \frac{\overline{C}_w'}{\sigma\sqrt{a'}}\left[\sinh^{-1}\left(\frac{2a' + b'm}{m\sqrt{\Delta'}}\right) - \sinh^{-1}\left(\frac{2a' + b'm_{stage2}}{m_{stage2}\sqrt{\Delta'}}\right)\right]
$$
$$
- \frac{\lambda_h}{\sigma^2 T_1\sqrt{c}}\left[\sinh^{-1}\left(\frac{2cm_{stage1} + b}{\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2cm_0 + b}{\sqrt{\Delta}}\right)\right]
$$
$$
- \frac{\lambda_h}{\sigma^2 T_2\sqrt{c'}}\left[\sinh^{-1}\left(\frac{2c'm + b'}{\sqrt{\Delta'}}\right) - \sinh^{-1}\left(\frac{2c'm_{stage2} + b'}{\sqrt{\Delta'}}\right)\right] \quad (4.58)
$$

$$
h = h_0 + \frac{g_s(m_{stage1}^2 - m_0^2)}{2(\sigma T_1)^2} + \frac{g_s(m^2 - m_{stage2}^2)}{2(\sigma T_2)^2}
$$
$$
+ \frac{mw}{\sigma T_2} - \frac{m_0 w_0}{\sigma T_1}
$$
$$
+ \left(\frac{m_{stage2}}{\sigma T_2} - \frac{m_{stage2}}{\sigma T_2}\right)\left\{w_0 - \frac{g_s}{\sigma T_1}(m_{stage1} - m_0)\right.
$$
$$
- \frac{\overline{C}_w}{\sigma\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a + bm_{stage1}}{m_{stage1}\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2a + bm_0}{m_0\sqrt{\Delta}}\right)\right]
$$
$$
\left. - \frac{\lambda_h}{\sigma^2 T_1\sqrt{c}}\left[\sinh^{-1}\left(\frac{2cm_{stage1} + b}{\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2cm_0 + b}{\sqrt{\Delta}}\right)\right]\right\}
$$
$$
+ \frac{\lambda_h}{\sigma(\sigma T_2)^2 c'}\left[\sqrt{c'm^2 + b'm + a'} - \sqrt{c'm_{stage2}^2 + b'm_{stage2} + a'}\right]
$$
$$
+ \frac{\lambda_h}{\sigma(\sigma T_1)^2 c}\left[\sqrt{cm_{stage1}^2 + bm_{stage1} + a} - \sqrt{cm_0^2 + bm_0 + a}\right] \quad (4.59)
$$

These are the equations that result from linking the first stage subarc to the second stage subarc. These equations will be used to evaluate the states at a time after staging occurs when the initial time is before staging. The first-order correction terms will require the analytic solution for the states at any future time along the zeroth-order trajectory.

Chapter 5

First-Order Corrections

The use of the asymptotic expansion of the dynamic programming equation as discussed in Chapter 2 by the approximate optimal guidance scheme is an improvement over past analytic techniques whose guidance laws were limited to operate in the exoatmospheric region [6, 14]. The higher-order correction terms of the HJB expansion can be used to compensate for the effects of the atmospheric forces neglected in the exoatmospheric solution. The determination of the first-order correction to the zeroth-order control is the subject of this chapter. As noted before, the solution to the first-order optimization problem requires only the integration of quadratures, which can be evaluated quickly enough to permit this method to be implemented as a real-time guidance scheme. The correction to the Lagrange multipliers and thus the correction to the control is constructed in the following sections. Also derived are all the partial derivatives needed to evaluate the quadratures. The partial derivative chain rule is employed since the analytic solution is found in the Cartesian frame while the first-order forcing function, R_1, used to evaluate the quadratures is expressed in the wind axis frame. Recall that the angle-of-attack is the control variable and the aerodynamic coefficients are modelled as functions of the angle-of-attack. For this reason the perturbation dynamics are left expressed in the wind axes frame.

5.1 Correction to the Lagrange Multipliers

The higher-order terms of the optimal return function were presented in Eq. (2.26).

$$P_i(x, t) = -\int_t^{t_f} R_i^0 d\tau$$

By taking the partial derivative of this integral the correction term to the Lagrange multiplier can be calculated. Recall,

$$P_{i_x} = \frac{\partial P_i}{\partial x} = -\int_t^{t_f} \frac{\partial R_i}{\partial x} d\tau + R_i|_t \frac{\partial t}{\partial x} - R_i|_{t_f} \frac{\partial t_f}{\partial x} \qquad (2.39)$$

where the first-order forcing function was $R_1 = -P_{0_x} g_0$.

The first-order correction term for the Lagrange multipliers is used to determine the first-order expansion term of the control. By the first-order optimality condition, Eq. (2.30), the correction to the control is obtained.

$$u_1 = -(f_{uu} P_{0_x})^{-1} [P_{0_x} g_u + P_{1_x} f_u] \qquad (5.1)$$

5.2 The First-Order Forcing Function

For the launch problem as formulated in the wind axis frame, the first-order forcing function is

$$
R_1 = \frac{r_e}{h_s} \left\{ \lambda_V \left[\frac{D}{m} - g_s \frac{r_e(2r_e + h)}{(r_e + h)^2} \sin\gamma + \frac{npA_e}{m} \cos\alpha \right] \right.
$$
$$
\left. - \frac{\lambda_\gamma}{V} \left[\frac{L}{m} + \left(\frac{V^2}{r} + g_s \frac{r_e(2r_e + h)}{(r_e + h)^2} \right) \cos\gamma - \frac{npA_e}{m} \sin\alpha \right] \right\} \qquad (5.2)
$$

The Lagrange multiplier for the first-order term of the expansion series is found by integrating the partial derivative of R_1 with respect to the initial state. For the launch problem, the optimal control depends on the Lagrange multipliers

for the velocity and flight path angle, i.e., $x = [V_0, \gamma_0]$. The partial derivative of the first-order forcing function with respect to the initial state is

$$
\begin{aligned}
\frac{\partial R_1}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\rho SV^2 r_e}{2mh_s}\right)\left(\lambda_V C_D - \frac{\lambda_\gamma}{V}C_L\right) \\
&+ \frac{\rho SV^2 r_e}{2mh_s}\left(\lambda_V \frac{\partial C_D}{\partial x} - \frac{\lambda_\gamma}{V}\frac{\partial C_L}{\partial x} + \frac{\partial \lambda_V}{\partial x}C_D - \frac{\partial}{\partial x}\left(\frac{\lambda_\gamma}{V}\right)C_L\right) \\
&- g_s \frac{h(2r_e + h)r_e}{h_s(r_e + h)^2}\left(\lambda_V \cos\gamma - \frac{\lambda_\gamma}{V}\sin\gamma\right)\frac{\partial \gamma}{\partial x} \\
&- g_s \frac{h(2r_e + h)r_e}{h_s(r_e + h)^2}\left[\frac{\partial \lambda_V}{\partial x}\sin\gamma + \frac{\partial}{\partial x}\left(\frac{\lambda_\gamma}{V}\right)\cos\gamma\right] \\
&- \frac{\partial}{\partial x}\left(g_s\frac{h(2r_e + h)r_e}{h_s(r_e + h)^2}\right)\left(\lambda_V \sin\gamma + \frac{\lambda_\gamma}{V}\cos\gamma\right) \\
&+ \frac{npA_e r_e}{mh_s}\left[\frac{\partial \lambda_V}{\partial x}\cos\alpha + \frac{\partial}{\partial x}\left(\frac{\lambda_\gamma}{V}\right)\sin\alpha\right] \\
&- \frac{npA_e r_e}{mh_s h_p}\left(\frac{\lambda_\gamma}{V}\sin\alpha + \lambda_V \cos\alpha\right)\frac{\partial h}{\partial x} \\
&- \frac{Vr_e}{h_s(r_e + h)}\left(\frac{\partial \lambda_\gamma}{\partial x}\cos\gamma - \lambda_\gamma \sin\gamma\frac{\partial \gamma}{\partial x}\right) \\
&- \frac{\partial}{\partial x}\left(\frac{Vr_e}{h_s(r_e + h)}\right)(\lambda_\gamma \cos\gamma)
\end{aligned}
\tag{5.3}
$$

where

$$
\frac{\partial}{\partial x}\left(\frac{\rho SV^2 r_e}{2mh_s}\right) = \frac{\rho SV^2 r_e}{2mh_s}\left[\frac{2}{V}\frac{\partial V}{\partial x} - \frac{1}{h_s}\frac{\partial h}{\partial x}\right]
\tag{5.4}
$$

$$
\frac{\partial}{\partial x}\left(g_s\frac{h(2r_e + h)r_e}{h_s(r_e + h)^2}\right) = \frac{2g_s r_e^3}{h_s(r_e + h)^3}\frac{\partial h}{\partial x}
\tag{5.5}
$$

$$
\frac{\partial}{\partial x}\left(\frac{Vr_e}{h_s(r_e + h)}\right) = \frac{Vr_e}{h_s(r_e + h)}\left[\frac{1}{V}\frac{\partial V}{\partial x} - \frac{1}{(r_e + h)}\frac{\partial h}{\partial x}\right]
\tag{5.6}
$$

$$
\frac{\partial C_D(M, \alpha)}{\partial x} = \frac{\partial C_D}{\partial M}\frac{\partial M}{\partial x} + \frac{\partial C_D}{\partial \alpha}\frac{\partial \alpha}{\partial x}
\tag{5.7}
$$

$$
\frac{\partial C_L(M, \alpha)}{\partial x} = \frac{\partial C_L}{\partial M}\frac{\partial M}{\partial x} + \frac{\partial C_L}{\partial \alpha}\frac{\partial \alpha}{\partial x}
\tag{5.8}
$$

The partials of the wind axis states and Lagrange multipliers are related to the partials of the analytic Cartesian states and Lagrange multipliers by the canonical coordinate transformation. These partial derivatives are presented in subsequent sections.

5.3 Relating the Partial Derivatives of the Wind Axis Frame to the Partial Derivatives of the Cartesian Frame

The canonical transformation of section 4.2 provides all the information needed to relate the analytic solution of the zeroth-order states and Lagrange multipliers to the states and Lagrange multipliers in the wind axis frame. Thus, the variations in the analytic Cartesian coordinates due to variations in the initial wind axis states can be determined and it was for this very reason the canonical transformation was necessary. Using the relationships obtained in section 4.2, the partial derivatives of the wind axis coordinates become

$$\frac{\partial V}{\partial x} = \frac{1}{V}\left(u\frac{\partial u}{\partial x} + w\frac{\partial w}{\partial x}\right) \tag{5.9}$$

$$\frac{\partial \gamma}{\partial x} = \tan\gamma\left[\left(\frac{1}{w} - \frac{w}{V^2}\right)\frac{\partial w}{\partial x} - \frac{u}{V^2}\frac{\partial u}{\partial x}\right] \tag{5.10}$$

$$\frac{\partial \lambda_V}{\partial x} = \frac{\partial \lambda_u}{\partial x}\cos\gamma - \frac{\partial \lambda_w}{\partial x}\sin\gamma - (\lambda_u\sin\gamma + \lambda_w\cos\gamma)\frac{\partial \gamma}{\partial x} \tag{5.11}$$

$$\frac{\partial \lambda_\gamma}{\partial x} = -V\left(\frac{\partial \lambda_u}{\partial x}\sin\gamma + \frac{\partial \lambda_w}{\partial x}\cos\gamma + [\lambda_u\cos\gamma - \lambda_w\sin\gamma]\frac{\partial \gamma}{\partial x}\right)$$
$$- \frac{\partial V}{\partial x}(\lambda_u\sin\gamma + \lambda_w\cos\gamma) \tag{5.12}$$

and from the zeroth-order control law Eq. (4.8)

$$\frac{\partial \alpha}{\partial x} = \cos \alpha \sin \alpha \left(\frac{1}{\lambda_\gamma} \frac{\partial \lambda_\gamma}{\partial x} - \frac{1}{\lambda_V} \frac{\partial \lambda_V}{\partial x} - \frac{1}{V} \frac{\partial V}{\partial x} \right) \tag{5.13}$$

Now that the partial derivatives for the wind axis coordinates are expressed in terms of the partial derivatives of the Cartesian coordinates, the partial derivatives of the Cartesian coordinates with respect to the initial states are to be derived along the analytic zeroth-order trajectory.

5.4 Partial Derivatives of the Analytic Solution

In this section, the partial derivatives of the Cartesian coordinates are derived. The zeroth-order analytic trajectory is used to evaluate the integral of the partial of the forcing function R_1 from the initial time to the final time. For the sake of notational brevity, the following common terms and their partial derivatives are defined.

5.4.1 Partial Derivatives of Some Common Terms

The partial derivatives of the constants a,b,c, and $\overline{C}_w$ used to express the analytic state equations are

$$
\begin{aligned}
a &= C_u^2 + \overline{C}_w^2 \\
\frac{\partial a}{\partial x} &= 2C_u \frac{\partial C_u}{\partial x} + 2\overline{C}_w \frac{\partial \overline{C}_w}{\partial x} \\
b &= -\frac{2}{\sigma T} \lambda_h \overline{C}_w \\
\frac{\partial b}{\partial x} &= -\frac{2}{\sigma T} \left[\overline{C}_w \frac{\partial \lambda_h}{\partial x} + \lambda_h \frac{\partial \overline{C}_w}{\partial x} \right] \\
c &= \left(\frac{\lambda_h}{\sigma T} \right)^2
\end{aligned}
$$

$$(5.14)$$
$$(5.15)$$

$$\frac{\partial c}{\partial x} = \frac{2\lambda_h}{(\sigma T)^2}\frac{\partial \lambda_h}{\partial x} \tag{5.16}$$

$$\frac{\partial \overline{C_w}}{\partial x} = \frac{\partial C_w}{\partial x} + \frac{m_0}{\sigma T}\frac{\partial \lambda_h}{\partial x} \tag{5.17}$$

Recall that the function $\Delta = 4ac - b^2$, so the partial derivative is

$$\frac{\partial \Delta}{\partial x} = 4a\frac{\partial c}{\partial x} + 4c\frac{\partial a}{\partial x} - 2b\frac{\partial b}{\partial x} \tag{5.18}$$

Let the arguments of the inverse hyperbolic sine function be denoted

$$\Im_1(m) = \frac{2cm + b}{\sqrt{\Delta}} \qquad \Im_2(m) = \frac{2a + bm}{m\sqrt{\Delta}} \tag{5.19}$$

Thus the partial derivatives of the arguments are

$$\frac{\partial \Im_1}{\partial x} = \frac{1}{\sqrt{\Delta}}\left[2(m - \frac{a\Im_1}{\sqrt{\Delta}})\frac{\partial c}{\partial x} + (1 + \frac{b\Im_1}{\sqrt{\Delta}})\frac{\partial b}{\partial x} - 2c\frac{\Im_1}{\sqrt{\Delta}}\frac{\partial a}{\partial x}\right] \tag{5.20}$$

$$\frac{\partial \Im_2}{\partial x} = \frac{1}{m\sqrt{\Delta}}\left[2(1 - m\frac{c\Im_2}{\sqrt{\Delta}})\frac{\partial a}{\partial x} + m(1 + \frac{b\Im_1}{\sqrt{\Delta}})\frac{\partial b}{\partial x} - ma\frac{\Im_2}{\sqrt{\Delta}}\frac{\partial c}{\partial x}\right] \tag{5.21}$$

and by the partial derivative chain rule for a trignometric function, the partials of the inverse hyperbolic sine functions are

$$\frac{\partial}{\partial x}\left(\sinh^{-1}\Im_1\right) = \frac{1}{\sqrt{1 + \Im_1^2}}\frac{\partial \Im_1}{\partial x} \tag{5.22}$$

$$\frac{\partial}{\partial x}\left(\sinh^{-1}\Im_2\right) = \frac{1}{\sqrt{1 + \Im_2^2}}\frac{\partial \Im_2}{\partial x} \tag{5.23}$$

5.4.2 Partial Derivatives of the Analytic States

The general form of the state equations in Eqs. (4.41)-(4.43) is used to derive the partial derivatives of the states with respect to the initial velocity or flight path angle. Using the terms defined in the previous section and

simplifying the equations, the partial derivatives are

$$\frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - \frac{1}{\sigma\sqrt{a}}\left[\sinh^{-1}\Im_2(m) - \sinh^{-1}\Im_2(m_0)\right]\left(\frac{\partial C_u}{\partial x} - \frac{C_u}{2a}\frac{\partial a}{\partial x}\right)$$
$$- \frac{C_u}{\sigma\sqrt{a}}\left[\frac{1}{\sqrt{1+\Im_2^2(m)}}\frac{\partial\Im_2(m)}{\partial x} - \frac{1}{\sqrt{1+\Im_2^2(m_0)}}\frac{\partial\Im_2(m_0)}{\partial x}\right] \quad (5.24)$$

$$\frac{\partial w}{\partial x} = \frac{\partial w_0}{\partial x} - \frac{1}{\sigma\sqrt{a}}\left[\sinh^{-1}\Im_2(m) - \sinh^{-1}\Im_2(m_0)\right]\left(\frac{\partial\overline{C}_w}{\partial x} - \frac{\overline{C}_w}{2a}\frac{\partial a}{\partial x}\right)$$
$$- \frac{\overline{C}_w}{\sigma\sqrt{a}}\left[\frac{1}{\sqrt{1+\Im_2^2(m)}}\frac{\partial\Im_2(m)}{\partial x} - \frac{1}{\sqrt{1+\Im_2^2(m_0)}}\frac{\partial\Im_2(m_0)}{\partial x}\right]$$
$$- \frac{1}{\sigma^2 T\sqrt{c}}\left[\sinh^{-1}\Im_1(m) - \sinh^{-1}\Im_1(m_0)\right]\left(\frac{\partial\lambda_h}{\partial x} - \frac{\lambda_h}{2c}\frac{\partial c}{\partial x}\right)$$
$$- \frac{\lambda_h}{\sigma^2 T\sqrt{c}}\left[\frac{1}{\sqrt{1+\Im_1^2(m)}}\frac{\partial\Im_1(m)}{\partial x} - \frac{1}{\sqrt{1+\Im_1^2(m_0)}}\frac{\partial\Im_1(m_0)}{\partial x}\right] \quad (5.25)$$

$$\frac{\partial h}{\partial x} = \frac{\partial w_0}{\partial x}\frac{(m-m_0)}{\sigma T}$$
$$- m\frac{\lambda_h}{(\sigma T)^2\sqrt{c}}\left[\frac{1}{\sqrt{1+\Im_1^2(m)}}\frac{\partial\Im_1(m)}{\partial x} - \frac{1}{\sqrt{1+\Im_1^2(m_0)}}\frac{\partial\Im_1(m_0)}{\partial x}\right]$$
$$- \frac{m}{\sigma T\sqrt{a}}\left[\sinh^{-1}\Im_2(m) - \sinh^{-1}\Im_2(m_0)\right]\left(\frac{\partial\overline{C}_w}{\partial x} - \frac{\overline{C}_w}{2a}\frac{\partial a}{\partial x}\right)$$
$$- m\frac{\overline{C}_w}{\sigma T\sqrt{a}}\left[\frac{1}{\sqrt{1+\Im_2^2(m)}}\frac{\partial\Im_2(m)}{\partial x} - \frac{1}{\sqrt{1+\Im_2^2(m_0)}}\frac{\partial\Im_2(m_0)}{\partial x}\right]$$
$$+ m\frac{\lambda_h}{2\sigma(\sigma T)^2 C^{3/2}}\left[\sinh^{-1}\Im_1(m) - \sinh^{-1}\Im_1(m_0)\right]\frac{\partial c}{\partial x}$$
$$- \frac{m}{\sigma(\sigma T)^2\sqrt{c}}\left[\sinh^{-1}\Im_1(m) - \sinh^{-1}\Im_1(m_0)\right]\frac{\partial\lambda_h}{\partial x}$$
$$+ \frac{1}{\sigma(\sigma T)^2 c}\left[\sqrt{cm^2+bm+a} - \sqrt{cm_0^2+bm_0+a}\right]\left(\frac{\partial\lambda_h}{\partial x} - \frac{\lambda_h}{c}\frac{\partial c}{\partial x}\right)$$
$$+ \frac{\lambda_h}{2\sigma(\sigma T)^2 c}\left[\frac{m^2\frac{\partial c}{\partial x}+m\frac{\partial b}{\partial x}+\frac{\partial a}{\partial x}}{\sqrt{cm^2+bm+a}} - \frac{m_0^2\frac{\partial c}{\partial x}+m_0\frac{\partial b}{\partial x}+\frac{\partial a}{\partial x}}{\sqrt{cm_0^2+bm_0+a}}\right] \quad (5.26)$$

The initial velocity components expressed in terms of the wind axis states are

$$u_0 = V_0 \cos \gamma_0, \qquad w_0 = -V_0 \sin \gamma_0 \qquad (5.27)$$

and therefore the partial derivatives with respect to the initial velocity and flight path angle are

$$\begin{aligned}
\frac{\partial u_0}{\partial V_0} &= \cos \gamma_0 \\
\frac{\partial u_0}{\partial \gamma_0} &= -V_0 \sin \gamma_0 \\
\frac{\partial w_0}{\partial V_0} &= -\sin \gamma_0 \\
\frac{\partial w_0}{\partial \gamma_0} &= -V_0 \cos \gamma_0 \qquad (5.28)
\end{aligned}$$

These partial derivatives are valid for a point during the first or second stage of the trajectory with initial condition corresponding to that subarc. For a point on the second subarc with first stage initial conditions, the state equations which link the two subarcs must be used. Note also that these equations all depend on the partial derivatives of the constants, λ_h, C_u, C_w, and m_f which are unknown. The partial derivatives of the constants are dependent on the initial and final conditions of the two-point boundary value problem. Using the transversality condition

$$H_f = -\lambda_h w_f + C_u \frac{T_2}{m_f} \cos \theta_f + \lambda_w(t_f)(-\frac{T_2}{m_f} \sin \theta_f + g_s) = -1 \qquad (5.29)$$

the partial derivative of the Hamiltonian at the final time is

$$\begin{aligned}
\frac{\partial H_f}{\partial x} &= 0 = -w_f \frac{\partial \lambda_h}{\partial x} + g_s \left(\frac{\partial C_w}{\partial x} - \frac{(m_f - m_0)}{\sigma T_2} \frac{\partial \lambda_h}{\partial x} - \frac{\lambda_h}{\sigma T_2} \frac{\partial m_f}{\partial x} \right) \\
&+ \frac{T_2 (bm_f + a)}{2m_f^2 \sqrt{cm_f^2 + bm_f + a}} \frac{\partial m_f}{\partial x} - \frac{T_2}{2m_f} \left[\frac{m_f^2 \frac{\partial c}{\partial x} + m_f \frac{\partial b}{\partial x} + \frac{\partial a}{\partial x}}{\sqrt{cm_f^2 + bm_f + a}} \right] (5.30)
\end{aligned}$$

These results produce a system of four equations $(\frac{\partial h_f}{\partial x}, \frac{\partial u_f}{\partial x}, \frac{\partial w_f}{\partial x}, \frac{\partial H_f}{\partial x})$ linear in the four unknown partial derivatives: $\frac{\partial \lambda_h}{\partial x}, \frac{\partial C_u}{\partial x}, \frac{\partial C_w}{\partial x}$, and $\frac{\partial m_f}{\partial x}$. The partial derivatives of the four constants are determined by the solution of this linear system.

5.4.3 Solution to the Linear System of Unknown Partials

For the second stage subarc, the solution to the linear system of four unknown partial derivatives in the partial derivatives of the four transcendental equations is determined by the matrix equation

$$
\begin{bmatrix}
0 \\[2mm]
-\frac{\partial w_0}{\partial x}\frac{(m_f - m_0)}{\sigma T} \\[2mm]
-\frac{\partial u_0}{\partial x} \\[2mm]
-\frac{\partial w_0}{\partial x}
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial H_f}{\partial \lambda_h} & \frac{\partial H_f}{\partial C_u} & \frac{\partial H_f}{\partial C_w} & \frac{\partial H_f}{\partial m_f} \\[2mm]
\frac{\partial h_f}{\partial \lambda_h} & \frac{\partial h_f}{\partial C_u} & \frac{\partial h_f}{\partial C_w} & \frac{\partial h_f}{\partial m_f} \\[2mm]
\frac{\partial u_f}{\partial \lambda_h} & \frac{\partial u_f}{\partial C_u} & \frac{\partial u_f}{\partial C_w} & \frac{\partial u_f}{\partial m_f} \\[2mm]
\frac{\partial w_f}{\partial \lambda_h} & \frac{\partial w_f}{\partial C_u} & \frac{\partial w_f}{\partial C_w} & \frac{\partial w_f}{\partial m_f}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \lambda_h}{\partial x} \\[2mm]
\frac{\partial C_u}{\partial x} \\[2mm]
\frac{\partial C_w}{\partial x} \\[2mm]
\frac{\partial m_f}{\partial x}
\end{bmatrix}
\tag{5.31}
$$

The coefficients of the matrix are

$$
\frac{\partial H_f}{\partial \lambda_h} = -w_f - g_s\frac{(m_f - m_0)}{\sigma T} + \frac{(m_f - m_0)[C_w + \frac{\lambda_h}{\sigma T}(m_0 - m_f)]}{\sigma m_f\sqrt{cm_f^2 + bm_f + a}}
\tag{5.32}
$$

$$
\frac{\partial H_f}{\partial C_u} = -\frac{C_u T}{m_f\sqrt{cm_f^2 + bm_f + a}}
\tag{5.33}
$$

$$
\frac{\partial H_f}{\partial C_w} = g_s - \frac{[C_w + \frac{\lambda_h}{\sigma T}(m_0 - m_f)]T}{m_f\sqrt{cm_f^2 + bm_f + a}}
\tag{5.34}
$$

$$
\frac{\partial H_f}{\partial m_f} = \frac{T}{m_f^2}\sqrt{cm_f^2 + bm_f + a} - g_s\frac{\lambda_h}{\sigma^2 T} + \frac{\lambda_h[C_w + \frac{\lambda_h}{\sigma T}(m_0 - m_f)]}{\sigma m_f\sqrt{cm_f^2 + bm_f + a}}
\tag{5.35}
$$

$$
\begin{aligned}
\frac{\partial u_f}{\partial x} = {} & -\frac{1}{\sigma\sqrt{a}}\left[\sinh^{-1}\mathfrak{S}_2(m_f) - \sinh^{-1}\mathfrak{S}_2(m_0)\right]\left(\frac{\partial C_u}{\partial x} - \frac{C_u}{2a}\frac{\partial a}{\partial x}\right) \\[2mm]
& -\frac{C_u}{\sigma\sqrt{a}}\left[\frac{1}{\sqrt{1 + \mathfrak{S}_2^2(m_f)}}\frac{\partial \mathfrak{S}_2(m_f)}{\partial x} - \frac{1}{\sqrt{1 + \mathfrak{S}_2^2(m_0)}}\frac{\partial \mathfrak{S}_2(m_0)}{\partial x}\right]
\end{aligned}
\tag{5.36}
$$

$$
\frac{\partial w_f}{\partial x} = -\frac{1}{\sigma\sqrt{a}}\left[\sinh^{-1}\Im_2(m_f) - \sinh^{-1}\Im_2(m_0)\right]\left(\frac{\partial \overline{C}_w}{\partial x} - \frac{\overline{C}_w}{2a}\frac{\partial a}{\partial x}\right)
$$

$$
-\frac{\overline{C}_w}{\sigma\sqrt{a}}\left[\frac{1}{\sqrt{1+\Im_2^2(m_f)}}\frac{\partial \Im_2(m_f)}{\partial x} - \frac{1}{\sqrt{1+\Im_2^2(m_0)}}\frac{\partial \Im_2(m_0)}{\partial x}\right]
$$

$$
-\frac{1}{\sigma^2 T\sqrt{c}}\left[\sinh^{-1}\Im_1(m_f) - \sinh^{-1}\Im_1(m_0)\right]\left(\frac{\partial \lambda_h}{\partial x} - \frac{\lambda_h}{2c}\frac{\partial c}{\partial x}\right)
$$

$$
-\frac{\lambda_h}{\sigma^2 T\sqrt{c}}\left[\frac{1}{\sqrt{1+\Im_1^2(m_f)}}\frac{\partial \Im_1(m_f)}{\partial x} - \frac{1}{\sqrt{1+\Im_1^2(m_0)}}\frac{\partial \Im_1(m_0)}{\partial x}\right] \quad (5.37)
$$

$$
\frac{\partial h_f}{\partial x} = -m_f\frac{\lambda_h}{(\sigma T)^2\sqrt{c}}\left[\frac{1}{\sqrt{1+\Im_1^2(m_f)}}\frac{\partial \Im_1(m_f)}{\partial x} - \frac{1}{\sqrt{1+\Im_1^2(m_0)}}\frac{\partial \Im_1(m_0)}{\partial x}\right]
$$

$$
-\frac{m_f}{\sigma T\sqrt{a}}\left[\sinh^{-1}\Im_2(m_f) - \sinh^{-1}\Im_2(m_0)\right]\left(\frac{\partial \overline{C}_w}{\partial x} - \frac{\overline{C}_w}{2a}\frac{\partial a}{\partial x}\right)
$$

$$
-m_f\frac{\overline{C}_w}{\sigma T\sqrt{a}}\left[\frac{1}{\sqrt{1+\Im_2^2(m_f)}}\frac{\partial \Im_2(m_f)}{\partial x} - \frac{1}{\sqrt{1+\Im_2^2(m_0)}}\frac{\partial \Im_2(m_0)}{\partial x}\right]
$$

$$
+m_f\frac{\lambda_h}{2\sigma(\sigma T)^2 C^{3/2}}\left[\sinh^{-1}\Im_1(m_f) - \sinh^{-1}\Im_1(m_0)\right]\frac{\partial c}{\partial x}
$$

$$
-\frac{m}{\sigma(\sigma T)^2\sqrt{c}}\left[\sinh^{-1}\Im_1(m_f) - \sinh^{-1}\Im_1(m_0)\right]\frac{\partial \lambda_h}{\partial x}
$$

$$
+\frac{1}{\sigma(\sigma T)^2 c}\left[\sqrt{cm_f^2 + bm_f + a} - \sqrt{cm_0^2 + bm_0 + a}\right]\left(\frac{\partial \lambda_h}{\partial x} - \frac{\lambda_h}{c}\frac{\partial c}{\partial x}\right)
$$

$$
+\frac{\lambda_h}{2\sigma(\sigma T)^2 c}\left[\frac{m_f^2\frac{\partial c}{\partial x} + m_f\frac{\partial b}{\partial x} + \frac{\partial a}{\partial x}}{\sqrt{cm_f^2 + bm_f + a}} - \frac{m_0^2\frac{\partial c}{\partial x} + m_0\frac{\partial b}{\partial x} + \frac{\partial a}{\partial x}}{\sqrt{cm_0^2 + bm_0 + a}}\right] \quad (5.38)
$$

where the equations $\frac{\partial u_f}{\partial x}$, $\frac{\partial w_f}{\partial x}$, and $\frac{\partial h_f}{\partial x}$ are the same as derived for the analytic state partials but are derived with respect to the constant parameters, i.e. $x = \{\lambda_h, C_u, C_w\}$. All these terms thus depend on the partial derivatives of the common terms a, b, c with respect to the constant parameters. So,

$$
\frac{\partial \overline{C}_w}{\partial \lambda_h} = \frac{m_0}{\sigma T} \qquad \frac{\partial \overline{C}_w}{\partial C_w} = 1 \qquad \frac{\partial \overline{C}_w}{\partial C_u} = 0
$$

$$
\frac{\partial a}{\partial C_u} = 2C_u + 2\overline{C}_w\frac{\partial \overline{C}_w}{\partial C_w}
$$

$$\frac{\partial a}{\partial C_w} = 2\overline{C}_w \frac{\partial \overline{C}_w}{\partial C_w}$$

$$\frac{\partial a}{\partial \lambda_h} = 2\overline{C}_w \frac{\partial \overline{C}_w}{\partial \lambda_h}$$

$$\frac{\partial b}{\partial C_u} = 0 \qquad \frac{\partial b}{\partial C_w} = -\frac{2}{\sigma T}\lambda_h \frac{\partial \overline{C}_w}{\partial C_w}$$

$$\frac{\partial b}{\partial \lambda_h} = -\frac{2}{\sigma T}\left[\overline{C}_w + \lambda_h \frac{\partial \overline{C}_w}{\partial \lambda_h}\right]$$

$$\frac{\partial c}{\partial \lambda_h} = \frac{2\lambda_h}{(\sigma T)^2} \qquad \frac{\partial c}{\partial C_u} = 0 \qquad \frac{\partial c}{\partial C_w} = 0$$

Remember that the variation of the terms with respect to the final mass is also needed. For the arguments of the inverse hyperbolic sine functions, the partial derivatives with respect to the final mass become

$$\frac{\partial \Im_1(m_f)}{\partial m_f} = \frac{2c}{\sqrt{\Delta}}, \qquad \frac{\partial \Im_2(m_f)}{\partial m_f} = -\frac{2a}{m_f^2\sqrt{\Delta}}$$

The partial derivatives of the analytic states with respect to the final mass are

$$\frac{\partial u_f}{\partial m_f} = -\frac{C_u}{\sigma\sqrt{a}} \frac{1}{\sqrt{1 + \Im_2^2(m_f)}} \frac{\partial \Im_2(m_f)}{\partial x}$$

$$\frac{\partial w_f}{\partial m_f} = -\frac{g_s}{\sigma T} - \frac{\overline{C}_w}{\sigma\sqrt{a}} \frac{1}{\sqrt{1 + \Im_2^2(m_f)}} \frac{\partial \Im_2(m_f)}{\partial m_f}$$

$$- \frac{\lambda_h}{\sigma^2 T\sqrt{c}} \frac{1}{\sqrt{1 + \Im_1^2(m_f)}} \frac{\partial \Im_1(m_f)}{\partial m_f}$$

$$\frac{\partial h_f}{\partial m_f} = -\frac{g_s}{(\sigma T)^2}(m_f - m_0) + \frac{w_0}{\sigma T}$$

$$- \frac{\lambda_h}{\sigma(\sigma T)^2\sqrt{c}}\left[\sinh^{-1}\left(\frac{2cm_f + b}{\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2cm_0 + b}{\sqrt{\Delta}}\right)\right]$$

$$- \frac{\overline{C}_w}{\sigma(\sigma T)\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a + bm_f}{m_f\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2a + bm_0}{m_0\sqrt{\Delta}}\right)\right]$$

$$- m_f\frac{\lambda_h}{(\sigma T)^2\sqrt{c}} \frac{1}{\sqrt{1 + \Im_1^2(m_f)}} \frac{\partial \Im_1(m_f)}{\partial m_f}$$

$$-m_f \frac{\overline{C}_w}{\sigma T \sqrt{a}} \frac{1}{\sqrt{1 + \Im_2^2(m_f)}} \frac{\partial \Im_2(m_f)}{\partial m_f}$$

$$+ \frac{\lambda_h}{2\sigma(\sigma T)^2 c} \frac{(2cm_f + b)}{\sqrt{cm_f^2 + bm_f + a}}$$

All these relationships are used to determine the coefficient terms of the algebraic set of equations. The variations in the constant parameters of the zeroth-order two-point boundary value problem with respect to variations in the initial states can subsequently be determined. These variations are embedded in the quadratures used to calculate the first-order correction to the Lagrange multipliers and determine how a change in the initial conditions changes the path while flying along a path which will satisfy the terminal boundary conditions.

For the situation where the vehicle has not yet staged, the partial derivatives are similar to those shown above but the equations of section 4.4 which link the two subarcs of the trajectory are used.

Chapter 6

Aerodynamic Effect along the Zeroth-Order Trajectory

Previously the problem of minimizing the fuel into orbit for the flight of a rocket in a vacuum over a flat nonrotating Earth was the zeroth-order problem, i.e., $\varepsilon = 0$. It was found that this zeroth-order trajectory deviated significantly from the optimal trajectory and the resulting correction terms were not small as was assumed in deriving the expansion method. To compensate for this problem the zeroth-order trajectory needs to be reshaped in order to keep the assumed perturbing effects small. One method that might work is to include a constraint on the control which will limit the zeroth-order angle-of-attack and thus the aerodynamics generated along the zeroth-order path. The problem in implementing such a constraint is that the zeroth-order solution must still be analytic. Since the analytic solution was found in the local horizon coordinate system the control was the pitch angle. From the standpoint of the physics of the problem, there is no logical constraint which can be imposed on the pitch angle. Limiting the angle-of-attack would create a mixed constraint in the local horizon coordinate frame involving the state and the control and this type of constraint is difficult to solve. A practical and necessary constraint for launching a rocket is a dynamic pressure limit. How such a constraint may be incorporated theoretically in the HJB-PDE expansion technique is presented in appendix[C]. But a dynamic pressure constraint arc also does not allow an analytic solution to the zeroth-order problem. Therefore, the zeroth-order

trajectory was modulated by including aerodynamic terms in the zeroth-order problem formulation. This process involved averaging the aerodynamics along the vacuum trajectory and solving anew the zeroth-order two-point boundary value problem. This technique was suggested by the successive approximation method used in [15]. By modelling the aerodynamics as constant terms, closed form solutions are still available. This chapter presents the details of including aerodynamic pulse functions averaged in the local horizon and body axes coordinate systems.

6.1 Inclusion of an Aerodynamic Effect in the Zeroth-Order Problem

Instead of assuming flight in a vacuum, the zeroth-order problem is now formulated to include aerodynamic terms. Then if $\varepsilon = 0$ the equations of motion for the zeroth-order problem, valid over both subarcs, become

$$
\begin{aligned}
\dot{h} &= V \sin \gamma \\
\dot{V} &= \frac{T}{m} \cos \alpha - g_s \sin \gamma + \frac{\mathcal{D}}{m} \\
\dot{\gamma} &= \frac{T}{mV} \sin \alpha - \frac{g_s}{V} \cos \gamma - \frac{\mathcal{L}}{mV} \\
\dot{\theta} &= \frac{V \cos \gamma}{r_e} \\
\dot{m} &= -\sigma T \Longrightarrow m = m_0 - \sigma T (\tau - \tau_0)
\end{aligned}
\tag{6.1}
$$

where

$$
\begin{aligned}
\mathcal{D} &= \left(A_x^0 \cos \gamma - A_z^0 \sin \gamma \right) \\
\mathcal{L} &= \left(A_x^0 \sin \gamma + A_z^0 \cos \gamma \right)
\end{aligned}
\tag{6.2}
$$

are the assumed lift and drag forces along the zeroth-order trajectory. The constant terms A_x^0, A_z^0 are the averaged aerodynamic forces in the x- and z-directions. For a vacuum zeroth-order trajectory these terms would be identically zero. Nonzero values will be used in order to improve the zeroth-order trajectory and keep the perturbation effect due to the neglected aerodynamics relatively small compared to the effects due to thrust and gravity. Since these terms are added to the zeroth-order dynamics, identical terms of opposite sign are included in the perturbation dynamics. Thus their effect is identically zero in the full-order system of equations.

The variational Hamiltonian is altered by the inclusion of these terms, e.g.

$$
\begin{aligned}
H \;=\; & -\lambda_h V \sin\gamma + \lambda_V(\frac{T}{m}\cos\alpha - g_s \sin\gamma + \frac{\mathcal{D}}{m}) \\
& +\frac{\lambda_\gamma}{V}(\frac{T}{m}\sin\alpha - g_s \sin\gamma - \frac{\mathcal{L}}{m})
\end{aligned}
\tag{6.3}
$$

Notice since the pulse functions used in the aerodynamic terms are constants, the zeroth-order control law determined by the optimality condition is not changed from the solution obtained for vacuum flight.

$$
\tan\alpha = \frac{\lambda_\gamma}{V\lambda_V}
\tag{6.4}
$$

Once again the analytic solution to the zeroth-order problem will be found in the Cartesian coordinate system.
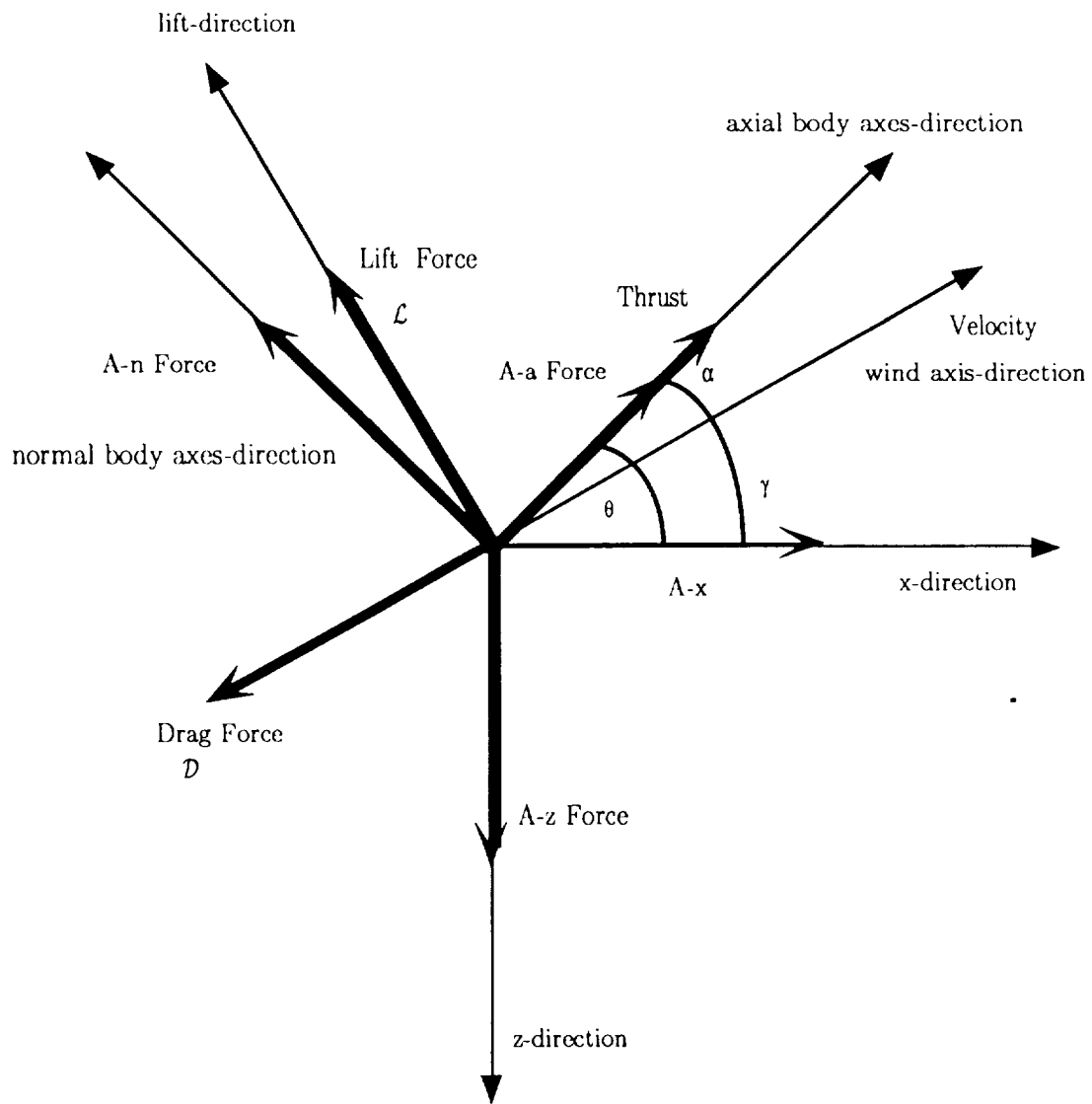
Figure 6.1: Coordinate frames for the aerodynamic pulse functions

6.1.1 Zeroth-Order Aerodynamic Effect in the Rectangular Coordinate System

The equations of motion in a Cartesian coordinate frame become

$$
\begin{aligned}
\dot{h} &= -w \\
\dot{w} &= -\frac{T}{m}\sin\theta_p + g_s + \frac{A_z^0}{m} \\
\dot{u} &= \frac{T}{m}\cos\theta_p + \frac{A_x^0}{m}
\end{aligned}
\tag{6.5}
$$

where the control variable for this problem becomes the pitch attitude $\theta_p = \alpha + \gamma$. The terms A_x^0 and A_z^0 represent the constant assumed aerodynamic forces along the zeroth-order trajectory in the x- and z-directions, respectively.

$$
\begin{aligned}
A_x^0 &= \frac{1}{t_i - t_{i+1}}\int_{t_i}^{t_{i+1}} A_x d\tau = \frac{1}{t_i - t_{i+1}}\int_{t_i}^{t_{i+1}} (-D\cos\gamma - L\sin\gamma)\, d\tau \\
A_z^0 &= \frac{1}{t_i - t_{i+1}}\int_{t_i}^{t_{i+1}} A_z d\tau = \frac{1}{t_i - t_{i+1}}\int_{t_i}^{t_{i+1}} (D\sin\gamma - L\cos\gamma)\, d\tau
\end{aligned}
\tag{6.6}
$$

Figures (6.2-6.3) show the aerodynamics averaged over a different number of intervals or subarcs.

The zeroth-order Hamiltonian is

$$
H = -\lambda_h w + \lambda_w\left(-\frac{T}{m}\sin\theta_p + g_s + \frac{A_x^0}{m}\right) + \lambda_u\left(\frac{T}{m}\cos\theta_p + \frac{A_z^0}{m}\right)
\tag{6.7}
$$

where λ_h, λ_w, and λ_u are Lagrange multipliers. These Lagrange multipliers are propagated by the Euler-Lagrange differential equation $\dot{\lambda}_y = -H_y^T$. Thus

$$
\dot{\lambda}_h = 0, \quad \dot{\lambda}_u = 0, \quad \dot{\lambda}_w = \lambda_h
\tag{6.8}
$$

with boundary conditions

$$
\lambda_h(\tau_f) = \nu_h, \quad \lambda_u(\tau_f) = \nu_u, \quad \lambda_w(\tau_f) = \nu_w
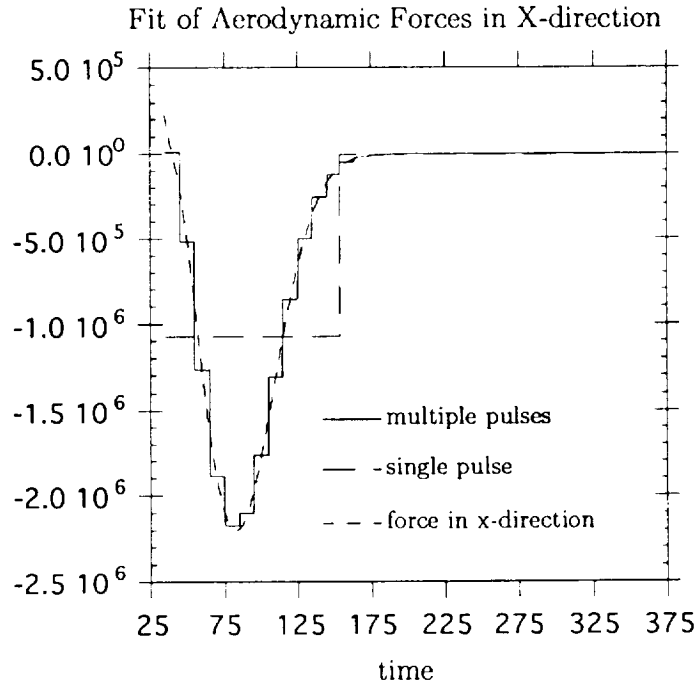\tag{6.9}
$$

Fit of Aerodynamic Forces in X-direction

Figure 6.2: Model for aerodynamic pulses in x-direction
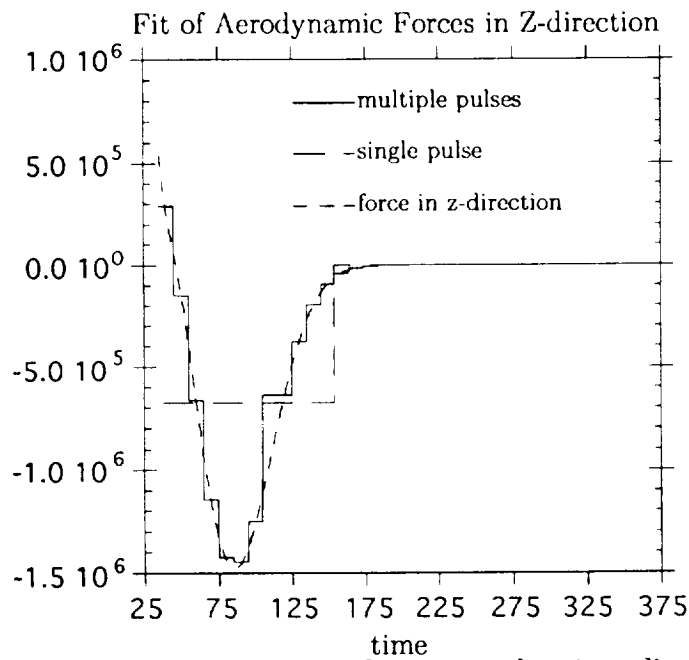
Fit of Aerodynamic Forces in Z-direction

Figure 6.3: Model for aerodynamic pulses in z-direction

where ν_h, ν_u, and ν_w are unknown Lagrange multipliers associated with the terminal constraints. Since the aerodynamic effect is added as a constant term there is no change in the solution to the Lagrange multipliers or to the control from the solution found for a vacuum zeroth-order trajectory. Therefore, the zeroth-order analytic state equations become

$$
\begin{aligned}
u = {} & u_0 - \frac{A_{i_x}^0}{\sigma T_i} \ln\left(\frac{m}{m_0}\right) \\
& - \frac{C_u}{\sigma\sqrt{a_i}}\left[\sinh^{-1}\left(\frac{2a_i + b_i m}{m\sqrt{\Delta_i}}\right) - \sinh^{-1}\left(\frac{2a_i + b_i m_0}{m_0\sqrt{\Delta}}\right)\right] \\
w = {} & w_0 - g_s\frac{(m - m_0)}{\sigma T_i} - \frac{A_{i_x}^0}{\sigma T_i}\ln\left(\frac{m}{m_0}\right) \\
& - \frac{\overline{C}_{w_i}}{\sigma\sqrt{a_i}}\left[\sinh^{-1}\left(\frac{2a_i + b_i m}{m\sqrt{\Delta_i}}\right) - \sinh^{-1}\left(\frac{2a_i + b_i m_0}{m_0\sqrt{\Delta_i}}\right)\right] \\
& - \frac{\lambda_h}{\sigma^2 T_i\sqrt{c_i}}\left[\sinh^{-1}(\frac{2c_i m + b_i}{\sqrt{\Delta_i}}) - \sinh^{-1}(\frac{2c_i m_0 + b_i}{\sqrt{\Delta_i}})\right] \\
h = {} & h_0 + w_0\frac{(m - m_0)}{(\sigma T_i)} - g_s\frac{(m - m_0)^2}{2(\sigma T_i)^2} \\
& + \frac{\lambda_h}{\sigma(\sigma T_i)^2 c_i}\left[(c_i m^2 + b_i m + a_i)^{1/2} - (c_i m_0^2 + b_i m_0 + a_i)^{1/2}\right] \\
& - \frac{\overline{C}_{w_i} m}{\sigma(\sigma T_i)\sqrt{a_i}}\left[\sinh^{-1}\left(\frac{2a_i + b_i m}{m\sqrt{\Delta_i}}\right) - \sinh^{-1}\left(\frac{2a_i + b_i m_0}{m_0\sqrt{\Delta_i}}\right)\right] \\
& - \frac{\lambda_h m}{\sigma(\sigma T_i)^2\sqrt{c_i}}\left[\sinh^{-1}(\frac{2c_i m + b_i}{\sqrt{\Delta_i}}) - \sinh^{-1}(\frac{2c_i m_0 + b_i}{\sqrt{\Delta_i}})\right] \\
& - \frac{A_{i_x}^0}{(\sigma T_i)^2}\left[m\ln\left(\frac{m}{m_0}\right) - m + m_0\right]
\end{aligned}
\tag{6.10}
$$

where

$$
\begin{aligned}
c_i &= (\tfrac{\lambda_h}{\sigma T_i})^2, \qquad b_i = -2\tfrac{\lambda_h}{\sigma T_i}\overline{C}_{w_i}, \qquad a_i = C_u^2 + \overline{C}_{w_i}^2 \\
\overline{C}_{w_1} &= C_w + \lambda_h\tfrac{m_0}{\sigma T_1}, \qquad \overline{C}_{w_2} = C_w + \lambda_h\tfrac{(m_0 - m_{s11})}{\sigma T_1} + \lambda_h\tfrac{m_{s12}}{\sigma T_2} \\
\Delta_i &= 4a_i c_i - b_i^2 = 4\left(\frac{C_u\overline{C}_{w_i}}{\sigma T_i}\right)^2 \qquad i = 1,2
\end{aligned}
\tag{6.11}
$$

and the subscript i refers to the current subarc. More pulse functions could be

used to model the aerodynamics in an attempt to capture the effect of the aero-dynamics in the closed form solution and thus the path would be broken up into smaller subarcs. Note that because the assumed aerodynamics are only con-stant terms their effect is an accumulative one. The zeroth-order trajectory is altered since the boundary conditions can not be satisfied flying the same path as the path flown in a vacuum. The vehicle does not modified its orientation instantaneously in order to reduce the aerodynamics that it will encounter, i.e. the vehicle cannot predict the aerodynamic effect on the vehicle by its choice of angle-of-attack. Thus any change is in the total energy of the system and the vehicle is not penalized for flying at large angles-of-attack and for incur-ring large drag forces. This can be seen in the new open loop zeroth-order trajectory in that the vehicle initially pitches over more than in the vacuum solution. But over the entire course of the trajectory the vehicle remains at lower angles-of-attack and does not lift up as much in the second stage. If more pulses are added the aerodynamics become larger over certain intervals and the vehicle reacts accordingly to these regions of large aerodynamic forces.

6.1.2 FIRST-ORDER CORRECTION TERMS

The correction terms to the zeroth-order problem can be calculated by the quadratures represented in (2.39). Therefore, for the launch problem

$$R_1 = \frac{r_e}{h_s} \left\{ \lambda_V \left[\frac{D + \mathcal{D}}{m} - g_s \frac{h(2r_e + h)}{(r_e + h)^2} \sin\gamma + \frac{npA_e}{m} \cos\alpha \right] \right. \tag{6.12}$$
$$\left. - \frac{\lambda_\gamma}{V} \left[\frac{L + \mathcal{L}}{m} + (\frac{V^2}{(r_e + h)} + g_s \frac{h(2r_e + h)}{(r_e + h)^2}) \cos\gamma - \frac{npA_e}{m} \sin\alpha \right] \right\}$$

The first-order term of the optimal return function evaluated along the zeroth-order trajectory with initial conditions before staging is written as in (2.26),

but separated into two integrals. Only the velocity and flight path angle state equations contain the control. Thus, the first-order terms in the expansion of the Lagrange multipliers associated with the velocity and flight path angle are the only co-state expansion terms needed to construct the first-order correction to the zeroth-order control. The partials of P_1 with respect to the arbitrary current conditions, $x = (V_0, \gamma_0)$, become

$$
\begin{aligned}
P_{1_x} &= \frac{\partial P_1}{\partial x} = -\int_t^{t_{stage}} \frac{\partial R_1(y_0^{opt})}{\partial x} d\tau \\
&\quad - \int_{t_{stage}}^{t_f} \frac{\partial R_1(y_0^{opt})}{\partial x} d\tau - R_1\left(y_0^{opt}(t_f)\right) \frac{\partial t_f}{\partial x}
\end{aligned}
\tag{6.13}
$$

Because aerodynamic pulses were added to the zeroth-order dynamics the opposite terms are added to the perturbation dynamics such that the overall system equations are unaltered. If the zeroth-order trajectory is the vacuum trajectory then the assumed aerodynamic terms $(\mathcal{D}, \mathcal{L})$ are zero. For nonzero assumed aerodynamic forces the new perturbing aerodynamic effect is the difference between the actual drag and the assumed drag along the zeroth-order path. It is necessary to keep this new perturbing aerodynamic effect small in order to accurately approximate the optimal solution. That is the entire reason for the inclusion of the aerodynamic pulse functions. The next sections present the results for various assumed aerodynamic pulses.

6.2 Results for the Rectangular Pulse Functions

It was found that the more pulses used the closer the first-order corrected solution came to the first-order solution obtained using a vacuum zeroth-order trajectory. The best solution for the approximated control was obtained

by using one pulse per stage. This seemed to keep the perturbing aerodynamic effect small over a larger span of the trajectory. The convergence of the Lagrange multipliers up to a first-order approximation using the one pulse aerodynamic functions for the zeroth-order problem is demonstrated by the plots presented in the Results chapter. Iteration of the zeroth-order trajectory for the assumed pulse functions was attempted but it was found that the first-order correction terms alternated back and forth between the optimal values and the solution based upon the vacuum zeroth-order path. This was a consequence of the assumed aerodynamics switching between large and small values on successive iterations. If large forces were assumed on a particular iteration than the actual aerodynamic forces along the new zeroth-order trajectory would become small and thus on the next iteration the assumed aerodynamic pulses would revert to smaller values and therefore the first-order corrections resembled the solutions obtained using a vacuum zeroth-order path. Attempts to average the iterations also proved unsatisfactory. For multiple pulses per stage, the averaged iterations did not adequate bring the assumed aerodynamic pulse functions closer to the actual forces along the new zeroth-order path. For a one pulse per stage solution the iterations could not improve on the solution obtained from the first iteration and thus were not worth the computational time and effort. In general, assuming more than one pulse per stage and more than one iteration caused the first-order corrections to go towards the values obtained assuming no aerodynamic forces along the zeroth-order trajectory. In a final attempt to lift the vehicle up and keep the vehicle from trying to pitch over, aerodynamic pulse functions were modelled as constants in the body-axes frame. The next section briefly describes that effort and the results.

6.3 Aero Pulses in the Body-Axes Frame

Because the use of aerodynamic pulses modelled as constant terms in the local horizon coordinate system the vehicle did not respond in an instantaneous fashion to the aerodynamics it encountered along a particular flight path. To remedy this situation the aerodynamic pulses were modelled as constant terms in the body-axes frame. Thus there are aerodynamic components tangent to and normal to the thrust. Rotation of these forces into the local horizon coordinate frame still allows an analytic solution to the zeroth-order problem but now the control law becomes a function of the aerodynamic effect assumed during a particular interval. This was not the case in using the aerodynamic pulses in the local horizon system as presented in the previous section. Because of the reliance of the zeroth-order control upon the aerodynamic pulses used, the control becomes discontinuous along the zeroth-order trajectory. Since the aerodynamic intervals are chosen as functions of a fixed time interval the Hamiltonian is also discontinuous across these intervals. The integrand used to derive the first-order correction to the Optimal Return Function and to the Lagrange multipliers is thus discontinuous and the integration of these terms along the zeroth-order path must be broken up according to the aerodynamic intervals. The equations of motion in rectangular coordinates for body-axes aerodynamic pulses are

$$
\begin{aligned}
\dot{h} &= -w \\
\dot{w} &= -\frac{T}{m}\sin\theta_p + g_s - \frac{A_A^0}{m}\sin\theta_p - \frac{A_N^0}{m}\cos\theta_p \\
\dot{u} &= \frac{T}{m}\cos\theta_p + \frac{A_A^0}{m}\cos\theta_p - \frac{A_N^0}{m}\sin\theta_p
\end{aligned}
\tag{6.14}
$$

The terms A_A^0 and A_N^0 represent the constant assumed aerodynamic forces along the zeroth-order trajectory in the axial and normal body-axes directions, respectively. The zeroth-order variational Hamiltonian is

$$H = -\lambda_h w + \lambda_w \left(-\frac{T + A_A^0}{m} \sin \theta_p + g_s - \frac{A_A^0}{m} \cos \theta_p\right) + \lambda_u \left(\frac{T + A_A^0}{m} \cos \theta_p - \frac{A_N^0}{m} \sin \theta_p\right)$$

$$(6.15)$$

The solution for the Lagrange multipliers does not change from the solution to the vacuum zeroth-order problem and the multipliers are continuous across subarc times, as are the states, since these times are considered fixed. The first-order optimality condition produces the following result.

$$\tan \theta_p = \frac{\lambda_w(T + A_A^0) + \lambda_u A_N^0}{\lambda_w A_N^0 - \lambda_u(T + A_A^0)} \tag{6.16}$$

Using this new control relationship in the state equations the closed form solution can still be obtained and the states are written as

$$
\begin{aligned}
u &= u_0 \\
&- \frac{C_u \tilde{T}}{\sigma T_i \sqrt{a_i}} \left[\sinh^{-1}\left(\frac{2a_i + b_i m}{m\sqrt{\Delta_i}}\right) - \sinh^{-1}\left(\frac{2a_i + b_i m_0}{m_0 \sqrt{\Delta}}\right)\right] \\
w &= w_0 - g_s \frac{(m - m_0)}{\sigma T_i} \\
&- \frac{\overline{C}_{w_i} \tilde{T}}{\sigma T_i \sqrt{a_i}} \left[\sinh^{-1}\left(\frac{2a_i + b_i m}{m\sqrt{\Delta_i}}\right) - \sinh^{-1}\left(\frac{2a_i + b_i m_0}{m_0 \sqrt{\Delta_i}}\right)\right] \\
&- \frac{\lambda_h \tilde{T}}{(\sigma T_i)^2 \sqrt{c_i}} \left[\sinh^{-1}\left(\frac{2c_i m + b_i}{\sqrt{\Delta_i}}\right) - \sinh^{-1}\left(\frac{2c_i m_0 + b_i}{\sqrt{\Delta_i}}\right)\right] \\
h &= h_0 + w_0 \frac{(m - m_0)}{(\sigma T_i)} - g_s \frac{(m - m_0)^2}{2(\sigma T_i)^2} \\
&+ \frac{\lambda_h \tilde{T}}{\sigma(\sigma T_i)^3 c_i} \left[(c_i m^2 + b_i m + a_i)^{1/2} - (c_i m_0^2 + b_i m_0 + a_i)^{1/2}\right] \\
&- \frac{\overline{C}_{w_i} \tilde{T} m}{(\sigma T_i)^2 \sqrt{a_i}} \left[\sinh^{-1}\left(\frac{2a_i + b_i m}{m\sqrt{\Delta_i}}\right) - \sinh^{-1}\left(\frac{2a_i + b_i m_0}{m_0 \sqrt{\Delta_i}}\right)\right]
\end{aligned}
$$

$$- \frac{\lambda_h \tilde{T} m}{(\sigma T_i)^3 \sqrt{c_i}} \left[\sinh^{-1}(\frac{2 c_i m + b_i}{\sqrt{\Delta_i}}) - \sinh^{-1}(\frac{2 c_i m_0 + b_i}{\sqrt{\Delta_i}}) \right] \qquad (6.17)$$

where

$$
\begin{aligned}
c_i &= (\tfrac{\lambda_h}{\sigma T_i})^2, \qquad b_i = -2 \tfrac{\lambda_h}{\sigma T_i} \overline{C}_{w_i}, \qquad a_i = C_u^2 + \overline{C}_{w_i}^2 \\
\overline{C}_{w_1} &= C_w + \lambda_h \tfrac{m_0}{\sigma T_1}, \qquad \overline{C}_{w_2} = C_w + \lambda_h \tfrac{(m_0 - m_{st1})}{\sigma T_1} + \lambda_h \tfrac{m_{st2}}{\sigma T_2} \\
\Delta_i &= 4 a_i c_i - b_i^2 = 4 \left(\frac{C_u \overline{C}_{w_i}}{\sigma T_i} \right)^2 \qquad i = 1, 2
\end{aligned}
\qquad (6.18)
$$

and the effective thrust $\tilde{T} = \sqrt{(T + A_A^0)^2 + (A_N^0)^2}$ is the magnitude of the sum of the thrust and assumed aerodynamic forces. A typical open loop zeroth-order trajectory is shown in figure (6.4). While the initial pitch over action was curtailed compared to the previous results, the trajectory still deviated from the optimal trajectory sharply especially in the regions of high dynamic pressure.

Corrections to the Lagrange Multipliers are made by the familiar equation

$$P_{1_x} = \frac{\partial P_1}{\partial x} = - \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \frac{\partial R_1(y_0^{\text{opt}})}{\partial x} d\tau - R_1 \left(y_0^{\text{opt}}(t_f) \right) \frac{\partial t_f}{\partial x} \qquad (6.19)$$

for n aerodynamic intervals and where

$$
\begin{aligned}
R_1 = \frac{r_e}{h_s} \Bigg\{ &\lambda_V \left[\frac{D + \mathcal{D}_b}{m} - g_s \frac{h(2 r_e + h)}{(r_e + h)^2} \sin \gamma + \frac{n p A_e}{m} \cos \alpha \right] \\
&- \frac{\lambda_\gamma}{V} \left[\frac{L - \mathcal{L}_b}{m} + (\frac{V^2}{(r_e + h)} + g_s \frac{h(2 r_e + h)}{(r_e + h)^2}) \cos \gamma - \frac{n p A_e}{m} \sin \alpha \right] \Bigg\}
\end{aligned}
\qquad (6.20)
$$

The assumed drag and lift terms are the transformation of the body-axes aerodynamic forces into the wind axes coordinate system, that is,

$$
\begin{aligned}
\mathcal{D}_b &= \left(A_A^0 \cos \alpha - A_N^0 \sin \alpha \right) \\
\mathcal{L}_b &= \left(A_A^0 \sin \alpha + A_N^0 \cos \alpha \right)
\end{aligned}
\qquad (6.21)
$$

The correction terms to the Lagrange multipliers based upon the zeroth-order trajectory using body-axes aerodynamic pulses did not give any improvement over the use of local horizon aerodynamic pulses. If anything the solutions obtained were worse since the trajectory was strongly influenced (as were the pulse functions) by the regions of high dynamic pressure and thus the perturbation aerodynamic effect remained large. The results from iterating with the averaged aerodynamic pulses and from averaging the iterations of the averaged pulses exhibited the same pattern as the local horizon case. Thus one pulse averaged over the first stage came closest to producing agreement with the optimal solution. The one positive effect of the body-axes approach when used in feedback to generate a trajectory was the elimination of the discontinuities in the control previously found when minimizing the Hamiltonian using the first stage aerodynamic model. Unfortunately, the path generated did not match as closely the optimal path as the results using the second stage aerodynamic model matched.
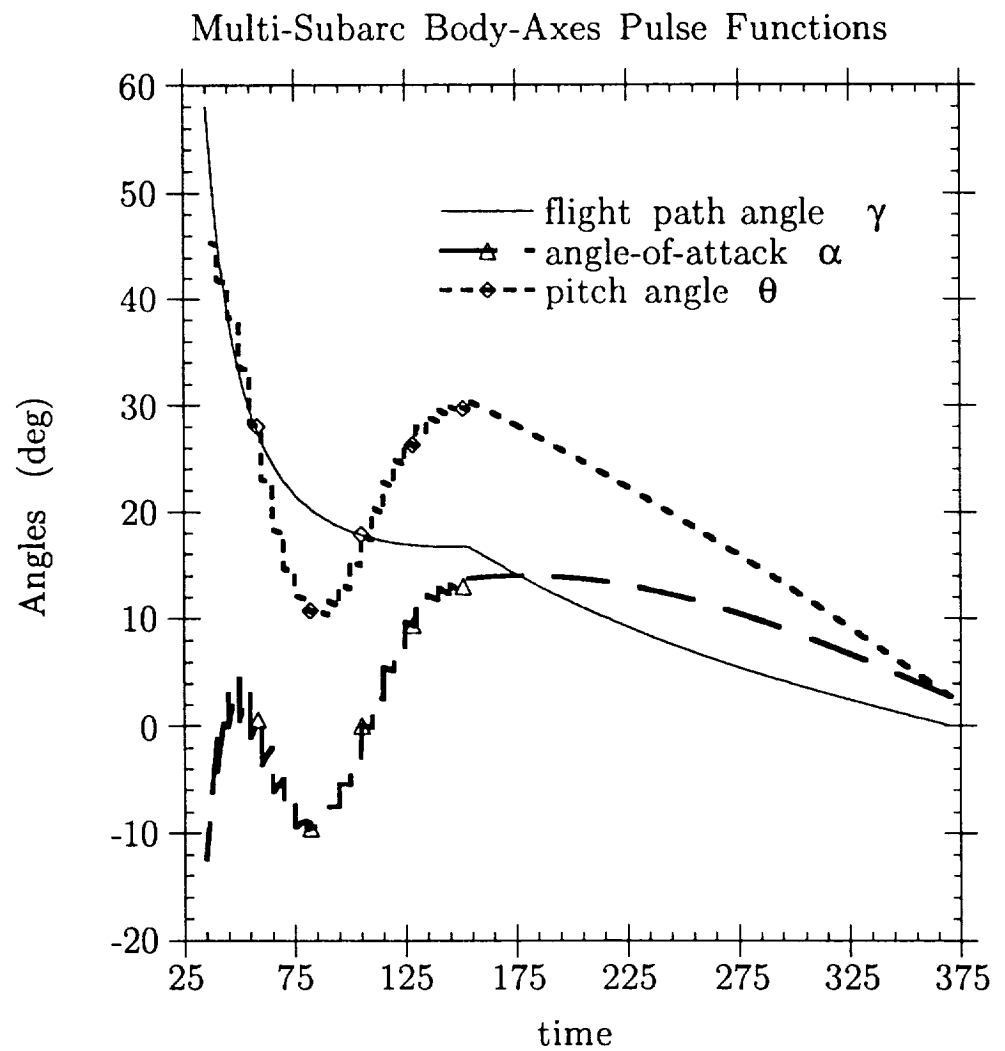
Multi-Subarc Body-Axes Pulse Functions

Figure 6.4: Open loop zeroth-order path for body-axes aerodynamic pulses

Chapter 7

Results

In this chapter the approximate optimal solution is compared to an optimal solution for the launch of a vehicle in the equatorial plane. While previous results for flight in the exoatmospheric regions [16] showed excellent matching of the approximate solution with the optimal, problems arose during the first stage. First, even at high altitudes where the aerodynamics are indeed perturbing effects to the vacuum trajectory, it was found that the linear control law derived for the first-order correction to the control (5.1) was in greater error than the error in the first-order corrected Lagrange multipliers. As a remedy the control was calculated by minimizing the Hamiltonian of the entire system using the Lagrange multipliers approximated to first-order. This produced the desired effect and the control profile converged to the solution obtained by the shooting method.

The next difficulty encountered was due to the first stage aerodynamic model. This model seemed to produce an irregular Hamiltonian. The Hamiltonian was badly behaved and exhibited discontinuities in the control at various points along the trajectory. The asymmetric configuration for the rocket and the cubic spline functions used to fit the aerodynamic data caused the Hamiltonian to take on almost identical values for different values of the angle-of-attack. This can be seen in figure (7.1) which are plots of the Hamilto-

nian versus the angle-of-attack at two consecutive points in the trajectory. The sequence shows the Hamiltonian exchanging the location of the minimum between positive and negative angles-of-attack. Part of the problem can be seen if the drag model is shown for larger angles-of-attack than was presented in chapter 3.3. Figure (7.2) shows the drag coefficient for different angles-of-attack and Mach numbers than would be encountered along the optimal trajectory. Remember the first-order correction terms are based on the aerodynamics along the vacuum path but the aerodynamics are not modelled adequately for these regions. The drag model of figure (7.2) shows the peculiar nature of the aerodynamics that would be used at the larger angles-of-attack of the zeroth-order trajectory. The smooth curve used to model the second stage aerodynamics was substituted into the algorithm to eliminate this strange behavior and remove the discontinuities in the control. This would prove successful. Figure (7.3) compares the drag and lift forces along the first stage of the open loop vacuum trajectory using the first and the second stage aerodynamic models. Another advantage of using the second stage aerodynamic model can be seen in that the drag has been reduced while the lift along the trajectory remains roughly the same.

Overcoming these difficulties still left a problem. The first-order correction exhibited a boundary layer type effect near the initial conditions. This would occur even if the problem was started at various points in the first stage. When the approximation method was used in feedback, this effect would diminish during the trajectory and the solution would converge to the optimal solution. In order to eliminate the initial over-corrections of the first-order approximation, the zeroth-order problem was reformulated to include an aero-
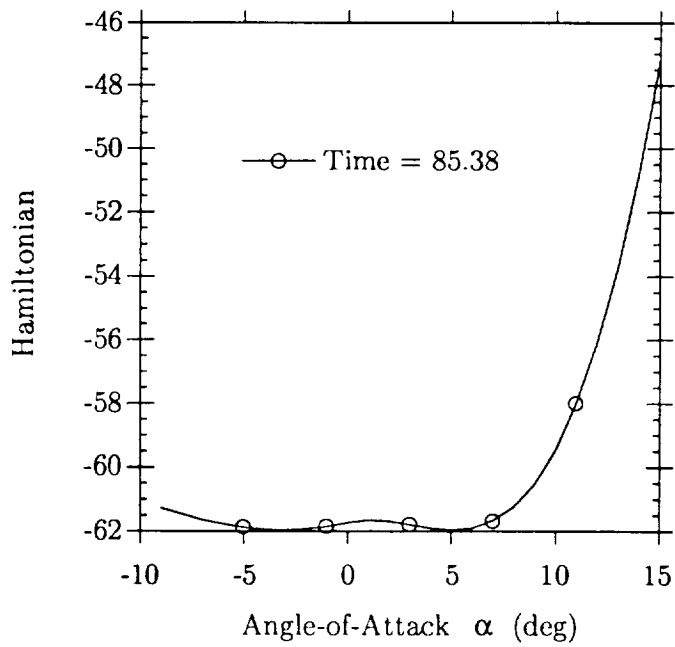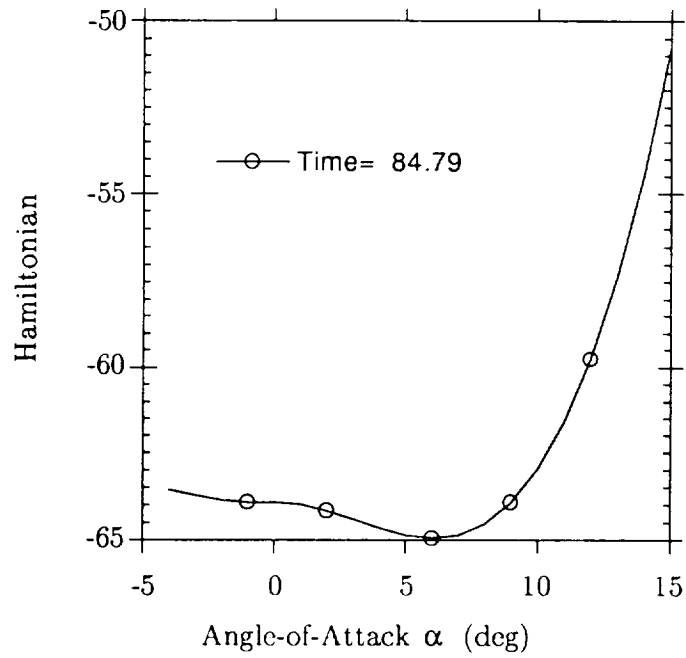
Figure 7.1: Hamiltonian versus Angle-of-Attack at continuous points of the first stage
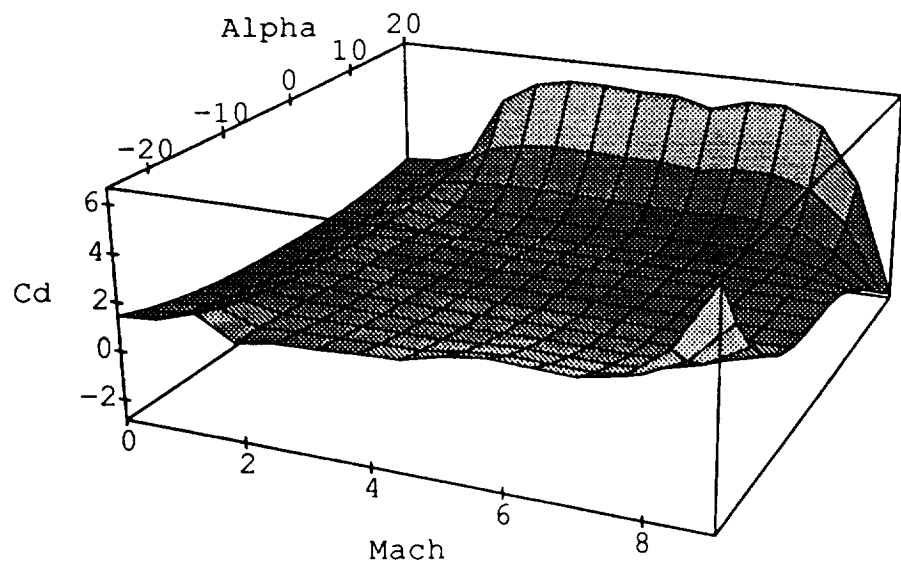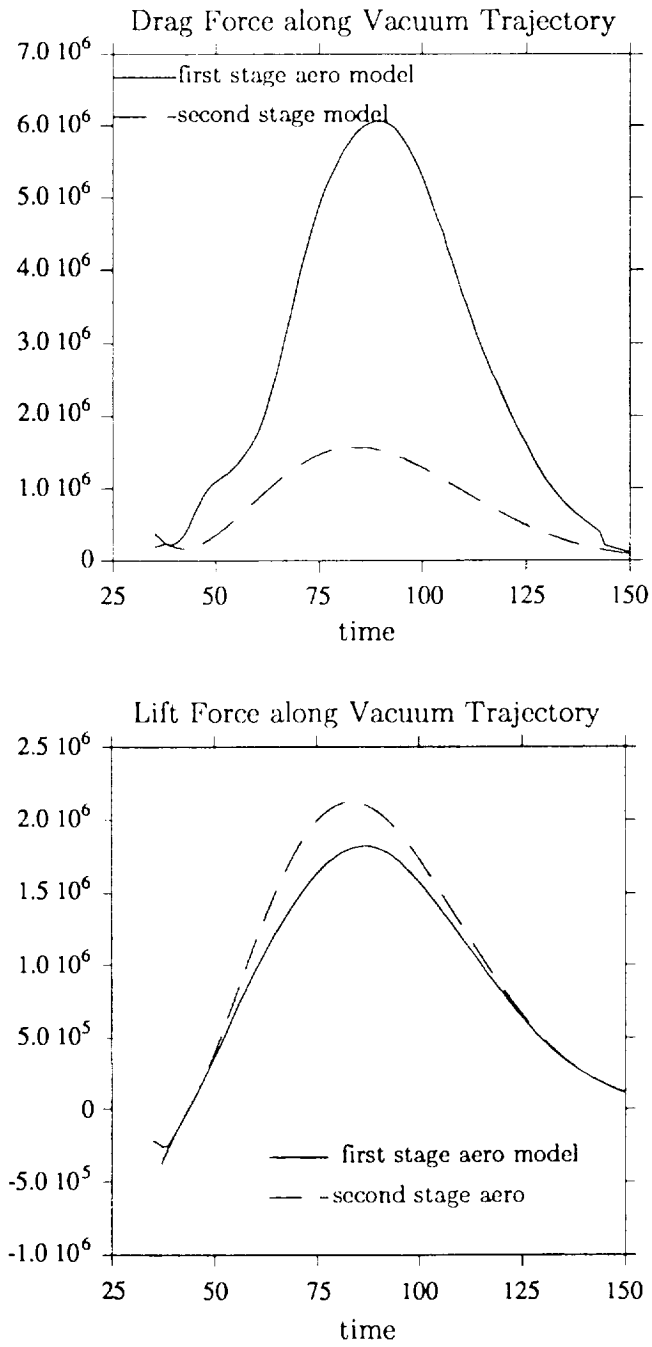
Figure 7.2: First stage model for the drag coefficient

Figure 7.3: Comparison of the first stage and second stage aero models along the vacuum path

Method	final time (sec.)	final weight (lbs.)	B.C. error	
			γ deg	h ft
zeroth order	371.50	322861.	-0.24	35.
first order	369.91	329293.	.03	-.002
first pulse	369.59	330576.	.0001	.0007
shooting	369.57	330678.	-	-

Table 7.1: Comparison of Results

dynamic effect. This technique was presented in chapter 6. In this chapter the results will be presented along with the results of the zeroth-order solution, the first-order solution without the aerodynamic effect in the zeroth-order problem, and the shooting method [17, 18].

The trajectories generated by the zeroth-order, the first-order with and without zeroth-order aerodynamic pulse functions, and the shooting method are shown in figures (7.4-7.9). Also plotted are the Lagrange multipliers for the closed loop trajectory, figures (7.10-7.11). Each technique ran on a IBM 3090 mainframe computer. Integration was done by an eighth-order Runge-Kutta method for the shooting method. The approximate optimal guidance schemes employed a fourth-order Runge-Kutta integrator. The approximate method used a fixed number of integration steps in the first and second stages with the control held fixed over each step. Four hundred steps were used in both the first and second stages. The time-to-stage was fixed at 153.54 seconds.

All the methods were started at the same initial conditions: $t_0 = 35$ sec., $h_0 = 660.$ ft., $V_0 = 9406.$ ft/s, $\gamma_0 = 58.$ deg., $m_0 = 3021107.44 lbs.$, $\theta_0 = -79.0$ deg., and $\chi = \phi = 0.0$ degrees. The terminal constraints to be satisfied are $h_f = 486080.$ ft., $V_f = 25770.$ ft/s, and $\gamma_f = 0.0$ degrees. The results are compared in Table (7.1). The solution shows the approximate optimal guidance law using the first-order correction term matches the control and state trajectories of the shooting method. Initially only the first-order correction with the aerodynamic pulse generates a nearly optimal trajectory. The cost obtained by the two techniques is nearly identical. The final weight using the shooting method was 330678. lbs. at a final time of 369.57 seconds. The final weight was 330576. lbs at a final time of 369.59 seconds when using the first-order approximation. The zeroth-order solution shows a greater variation in the control from the optimal control. The final weight obtained was 322861. lbs. at a final time of 371.5 seconds. The zeroth-order solution also does not satisfy all the boundary conditions as closely as the optimal and first-order solutions, with an error in the final flight path angle of -.24 degrees and an error in the final altitude of +40 feet. Because of this error in the terminal constraints, large angles-of-attack can be seen in fig. (7.4) for the zeroth-order solution in attempting to meet the terminal constraints. The first-order correction picked up most of the deviation of the zeroth-order trajectory from the optimal trajectory and as a result the boundary conditions are met more closely with a better behaved control. The most important aspect in obtaining good results is the convergence of the Lagrange multipliers to the optimal Lagrange multipliers. With the use of the aerodynamic pulses the flight path angle Lagrange multiplier approximated to first-order shows good
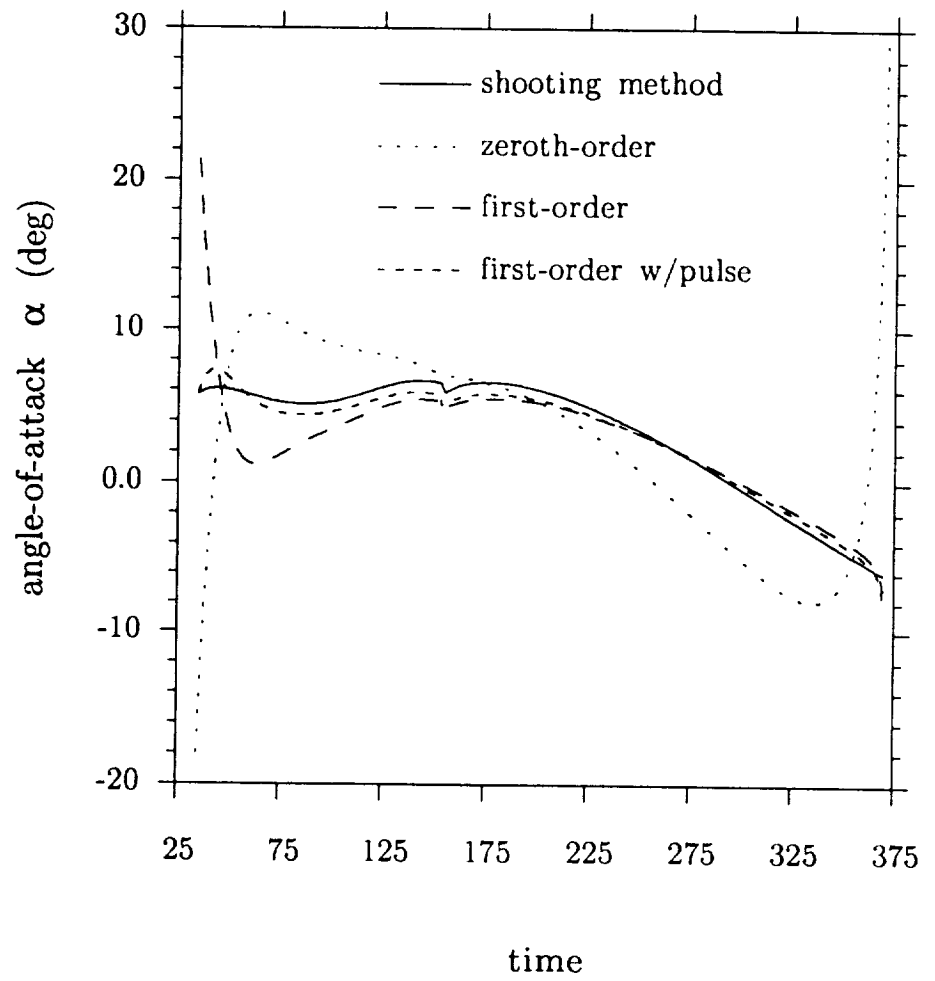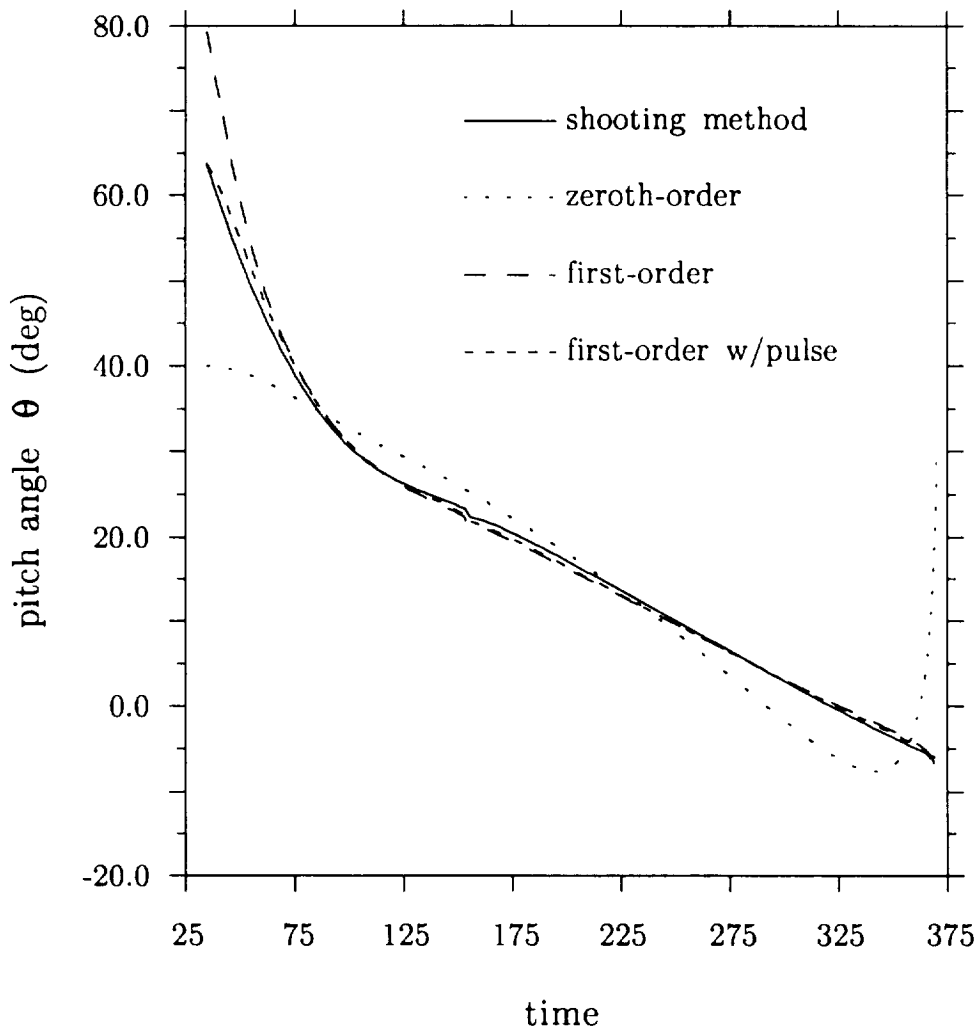
Figure 7.4: Angle-0f-Attack vs. Time

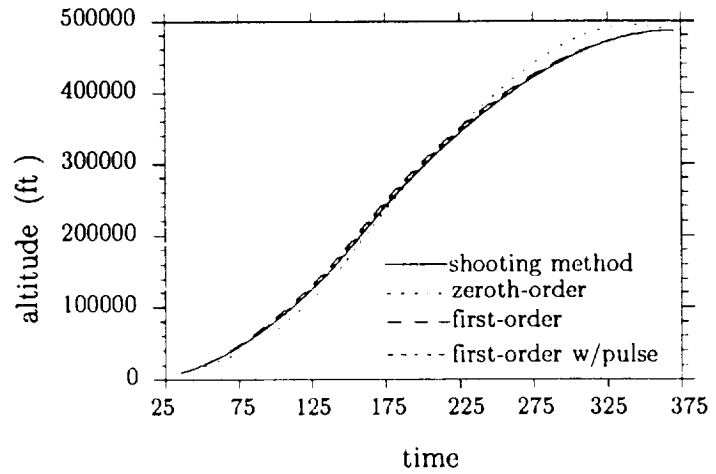Figure 7.5: Thrust Pitch Angle vs. Time

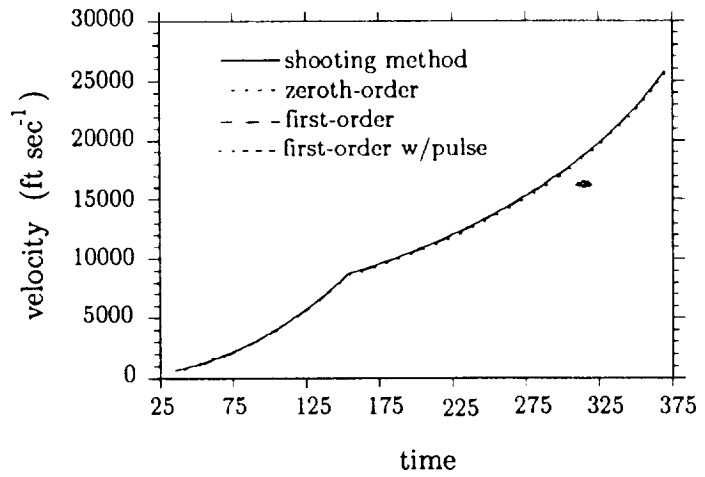Figure 7.6: Altitude vs. Time

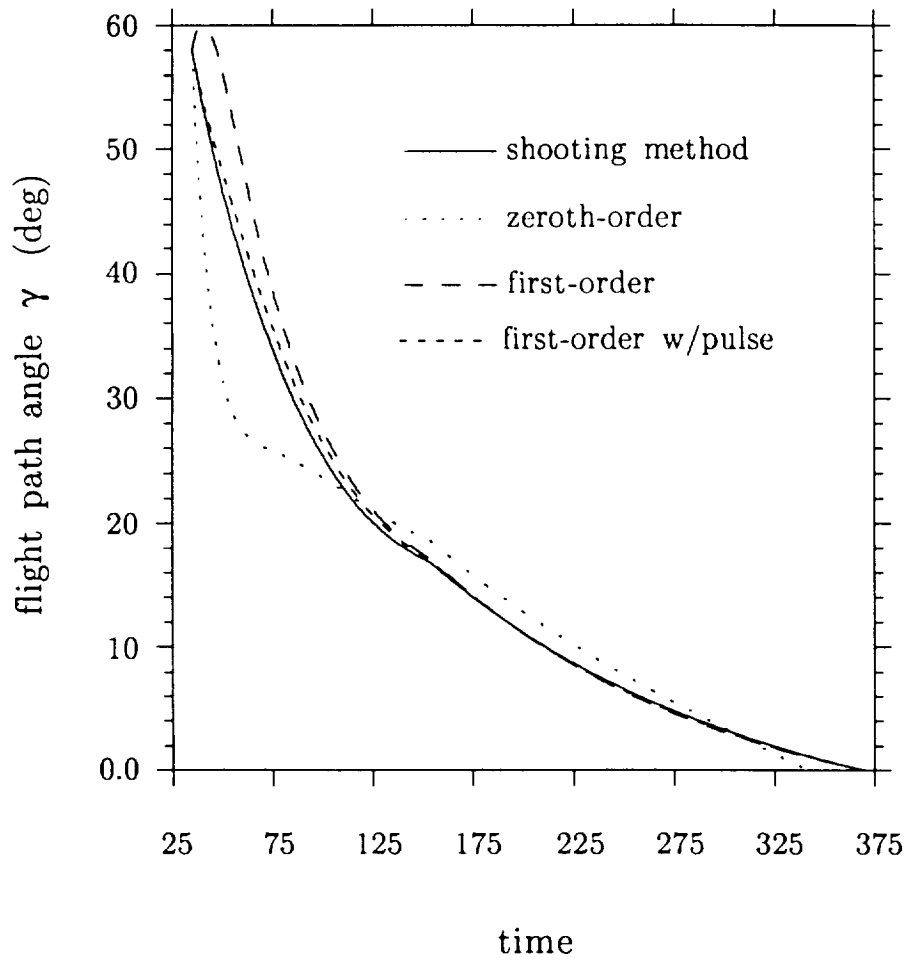

Figure 7.7: Velocity vs. Time
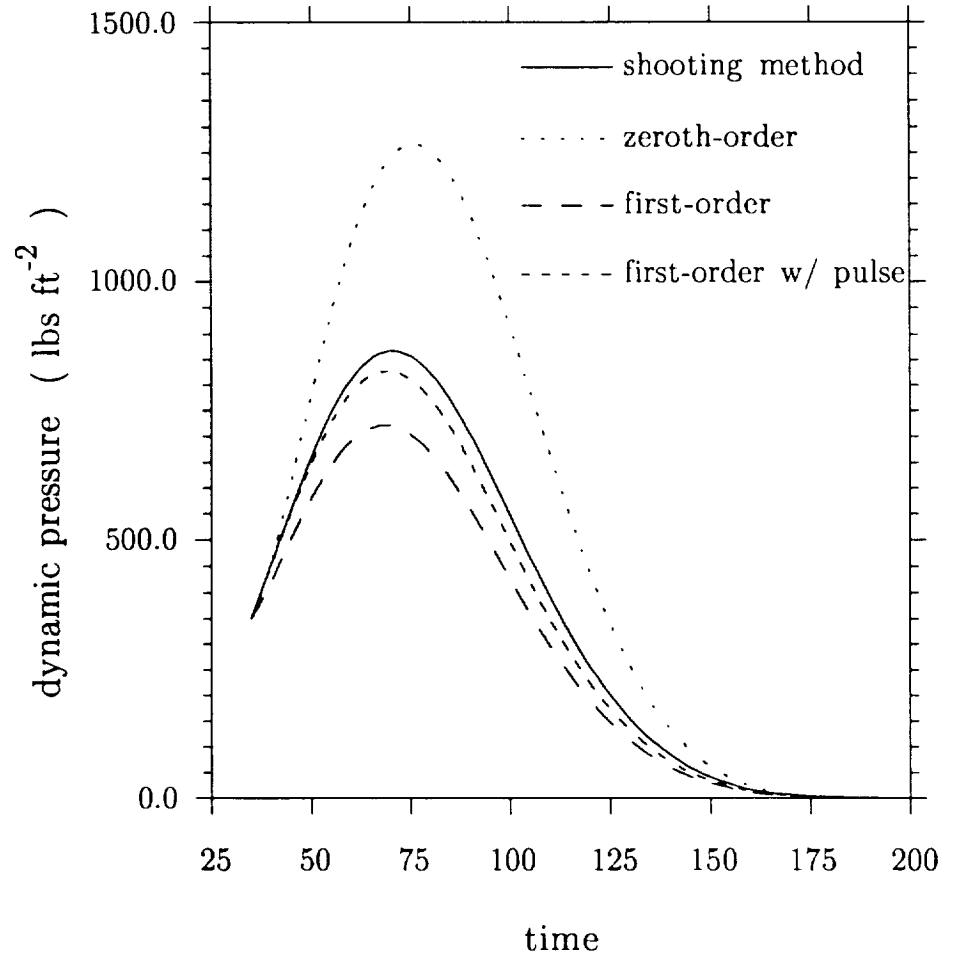
Figure 7.8: Flight Path Angle vs. Time
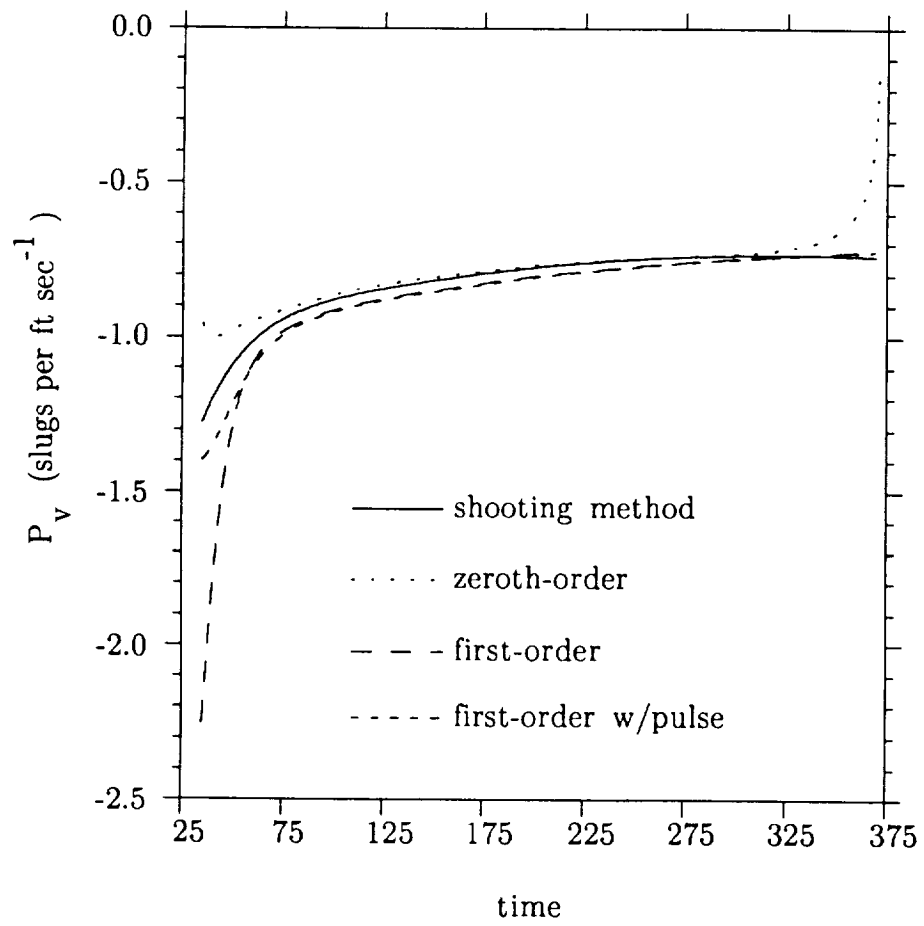
Figure 7.9: Dynamic Pressure vs. Time

Figure 7.10: Velocity Lagrange Multiplier vs. Time

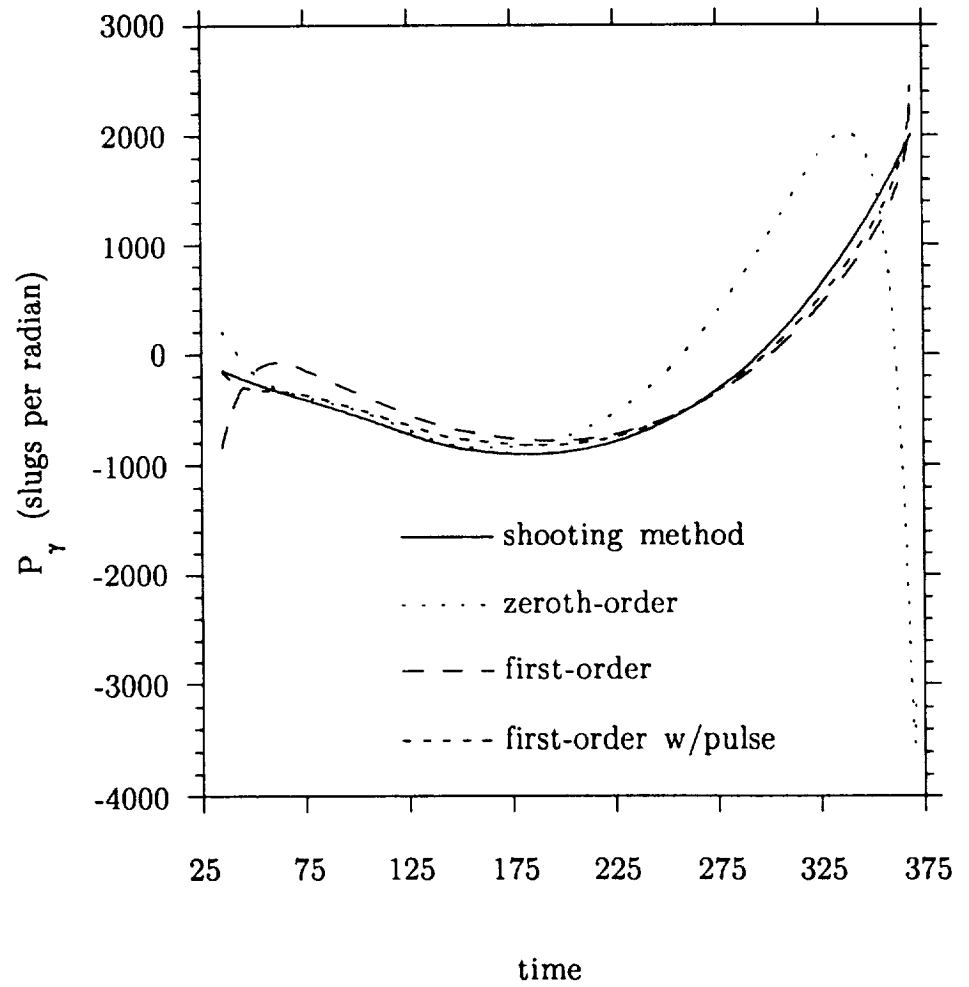Figure 7.11: Flight Path Lagrange Multiplier vs. Time

Method	zeroth order	first vacuum	first pulse	shooting
CPU time (sec)	49.	304.	344.	426.

Table 7.2: Comparison of computation time

agreement with the optimal solution. A last point about these result is that the inclusion of the rotation of the Earth in the problem is expected to continue to reduce the time of flight and consequently increase the final weight available at orbital insertion.

The convergence of the asymptotic expansion is indicated by the result of the first-order solution in comparison with the shooting method solution, thereby precluding the need to include higher-order correction terms. This convergence is tentative since it took the inclusion of the aerodynamic pulse functions in the zeroth-order problem to achieve the best results. Alas the convergence properties when using these pulses cannot be guaranteed or even quantified. Finally, since this algorithm is being proposed as a real-time guidance scheme the computational time that was needed to generate the entire trajectory by each method is presented in Table 7.2. While none of the codes have been optimized for computational efficiency, the use of quadratures does decrease the time needed to solve the launch problem in comparison to the shooting method. It should be noted that the flight time is approximately the same as the cpu time for the first-order approximation methods and that the shooting method was given a good initial guess (nearly converged) of the unknowns. As expected, the zeroth-order analytic solution was found extremely

quickly. The introduction of the aerodynamic pulse functions into the method caused a modest increase in the computation time.

Chapter 8

The Relationship between Calculus of Variations and the HJB equation

In this chapter the relationship between the Hamilton-Jacobi-Bellman approach and the calculus of variations approach is presented. First, the HJB expansion method is described in more detail in order to explicitly show the link between it and the perturbation of the canonical form of the Euler-Lagrange equations. The similarity of the terms involved and of the two solution techniques is shown. Next the solution of linear, first-order, partial differential equations is described. The significant result of the solution process is that the solution to a partial differential equation is equivalent to the solution of the characteristic curves represented by a set of ordinary differential equations. A simple derivation of the Lagrange multiplier differential equation from the HJB-PDE is also included. Lastly, the formulation of the ALS problem along with the results obtained when using the expansion of the calculus of variations method are presented.

8.1 Correction Terms to the Lagrange Multipliers

In Chapter 2 the equation for the Lagrange Multipliers (the change in the cost due to a change in the initial state) was determined to be

$$\frac{\partial P_1(x,t)}{\partial x} = -\int_t^{\tau_f} \frac{\partial R_1(y,\tau,P_{0_x})}{\partial x} d\tau - R_1|_{\tau_f} \frac{\partial \tau_f}{\partial x}$$

where the terms are evaluated along the zeroth-order path and the higher-order expansion terms $P_i(y_f, t_f) = 0$ $(i = 1, 2, \ldots)$ at the terminal boundary.

Now recall that the integrand terms were

$$R_1(y, \tau, P_{0_x}) = -P_{0_x}^T \left(g(y, u, \tau) + f_1(y, u, \tau) \right) \tag{8.1}$$

$$\frac{\partial R_1(y, \tau, P_{0_x})}{\partial x} = -\frac{\partial \left[P_{0_x}^T \left(g(y, u, \tau) + f_1(y, u, \tau) \right) \right]}{\partial x} \tag{8.2}$$

where the expansion term in the primary dynamics f_1 is

$$f_1(y, u, \tau) = \left. \frac{\partial f(y, u, \tau)}{\partial u} \right|_{\epsilon = 0} u_1 \tag{8.3}$$

Along the zeroth-order path the optimality condition $P_{0_x}^T f_u = 0$ eliminates the f_1 term in the integrand R_1

$$R_1(y, \tau, P_{0_x}) = -P_{0_x}^T g(y, u, \tau)$$

Therefore, the first-order term in the Lagrange multiplier expansion is

$$\frac{\partial P_1(x, t)}{\partial x} = -\int_t^{\tau_f} \left\{ g^T(y, u, \tau) \frac{\partial P_{0_x}}{\partial x} + P_{0_x}^T \left[\frac{\partial g}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} \right] \right\} d\tau$$
$$\left. - R_1 \right|_{\tau_f} \frac{\partial \tau_f}{\partial x} \tag{8.4}$$

Now the equation for the zeroth-order control can be written as a function of the independent variable, of the states, and of the partial derivative of the zeroth-order Optimal Return Function with respect to the states.

$$u_0(\tau) \Rightarrow P_{0_x}^T(\tau) f_u(y, u, \tau) = 0$$

Thus the variation of the control with respect to the initial state is

$$\frac{\partial u}{\partial x} \Rightarrow P_{0_x}^T(\tau)\frac{\partial f_u(y,u,\tau)}{\partial y}\frac{\partial y}{\partial x} + P_{0_x}^T(\tau)\frac{\partial f_u(y,u,\tau)}{\partial u}\frac{\partial u}{\partial x} + f_u^T(y,u,\tau)\frac{\partial P_{0_x}(\tau)}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} = -\left[P_{0_x}^T(\tau)\frac{\partial f_u}{\partial u}\right]^{-1}\left\{P_{0_x}^T(\tau)\frac{\partial f_u(y,u,\tau)}{\partial y}\frac{\partial y}{\partial x} + f_u^T(y,u,\tau)\frac{\partial P_{0_x}(\tau)}{\partial x}\right\}$$

Substituting this equation into the first-order Lagrange multiplier equation results in

$$\frac{\partial P_1(x,t)}{\partial x} = \int_t^{\tau_f}\left\{P_{0_x}^T\frac{\partial g}{\partial y} - P_{0_x}^T\frac{\partial g}{\partial u}\left[P_{0_x}^T\frac{\partial f_u(y,u,\tau)}{\partial u}\right]^{-1}P_{0_x}^T\frac{\partial f_u}{\partial y}\right\}\frac{\partial y}{\partial x}d\tau$$

$$- \int_t^{\tau_f}\left\{g^T(y,u,\tau) - P_{0_x}^T\frac{\partial g}{\partial u}\left[P_{0_x}^T\frac{\partial f_u(y,u,\tau)}{\partial u}\right]^{-1}f_u^T(y,u,\tau)\right\}\frac{\partial P_{0_x}}{\partial x}d\tau$$

$$- R_1\bigg|_{\tau_f}\frac{\partial \tau_f}{\partial x} \tag{8.5}$$

Note the notation used here is that the partial derivatives are taken with respect to an individual initial state x not the initial state vector.

The integrations thus depend on the variation of the zeroth-order states and the zeroth-order Lagrange multipliers due to variations in the initial state x, i.e. the terms $\frac{\partial y}{\partial x}$ and $\frac{\partial P_{0_x}}{\partial x}$ represent the state transition matrix. This matrix can be obtained from the zeroth-order analytic solution.

$$\frac{\partial y}{\partial x}(\tau) = \frac{\partial y}{\partial x}(t) + \int_t^\tau\left\{\frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right\}d\tau$$

Therefore the time derivative of $\frac{\partial y}{\partial x}$ is

$$\frac{d}{d\tau}\frac{\partial y}{\partial x}(\tau) = \left\{\frac{\partial f}{\partial y} - \frac{\partial f}{\partial u}\left[P_{0_x}^T(\tau)\frac{\partial f_u}{\partial u}\right]^{-1}P_{0_x}^T(\tau)\frac{\partial f_u(y,u,\tau)}{\partial y}\right\}\frac{\partial y}{\partial x}$$

$$- \frac{\partial f}{\partial u}\left[P_{0_x}^T(\tau)\frac{\partial f_u}{\partial u}\right]^{-1}f_u^T(y,u,\tau)\frac{\partial P_{0_x}}{\partial x}$$

where the equation for the control variation has been used. Similarly for the Lagrange Multipliers,

$$\frac{\partial P_{0_x}}{\partial x}(\tau) = \frac{\partial P_{0_x}}{\partial x}(t) - \int_t^T \left\{ P_{0_x}^T \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} \right) + f^T(y,u,\tau)\frac{\partial P_{0_x}}{\partial x} \right\} d\tau$$

$$\frac{d}{d\tau}\frac{\partial P_{0_x}}{\partial x}(\tau) = -\left\{ P_{0_x}^T\frac{\partial f}{\partial y} - P_{0_x}^T\frac{\partial f}{\partial u}\left[P_{0_x}^T(\tau)\frac{\partial f_u}{\partial u} \right]^{-1} P_{0_x}^T(\tau)\frac{\partial f_u(y,u,\tau)}{\partial y} \right\} \frac{\partial y}{\partial x}$$

$$- \left\{ f^T(y,u,\tau) - P_{0_x}^T\frac{\partial f_y}{\partial u}\left[P_{0_x}^T(\tau)\frac{\partial f_u}{\partial u} \right]^{-1} f_u^T(y,u,\tau) \right\} \frac{\partial P_{0_x}}{\partial x}$$

These coupled equations could be written in matrix notation as

$$\frac{d}{d\tau}\left[\begin{array}{c} \frac{\partial y}{\partial x} \\ \frac{\partial P_{0_x}}{\partial x} \end{array} \right] = \left[\begin{array}{cc} A_{yy} & A_{yP_{0_x}} \\ A_{P_{0_x}y} & A_{P_{0_x}P_{0_x}} \end{array} \right]\left[\begin{array}{c} \frac{\partial y}{\partial x} \\ \frac{\partial P_{0_x}}{\partial x} \end{array} \right]$$

where A_{yy}, $A_{yP_{0_x}}$, $A_{P_{0_x}y}$, $A_{P_{0_x}P_{0_x}}$ are the coefficients of the differential equations presented above.

The change in the parameters associated with the zeroth-order two-point boundary value problem due to a change in the initial states can be determined by the variation with respect to the initial states of the terminal boundary conditions and of the Hamiltonian at the final time. For every change in the initial state the transversality conditions must still be satisfied. So,

$$\frac{\partial \Psi_0}{\partial x}(y,\tau_f) = \frac{\partial \Psi_0}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial \Psi_0}{\partial \tau_f}\frac{\partial \tau_f}{\partial x} = 0$$

$$\frac{\partial P_{0_x}}{\partial x} = \frac{\partial \phi_{0_y}}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial \phi_{0_y}}{\partial \tau_f}\frac{\partial \tau_f}{\partial x} + \nu_0^T\frac{\partial \Psi_{0_y}}{\partial y}\frac{\partial y}{\partial x} + \nu_0^T\frac{\partial \Psi_{0_y}}{\partial \tau_f}\frac{\partial \tau_f}{\partial x} + \Psi_{0_y}^T\frac{\partial \nu_0}{\partial x}$$

Lastly,

$$Ham_0 = P_{0_x}^T f(y,u,\tau_f) = -\phi_{0_{\tau_f}} - \nu_0^T\Psi_{0_{\tau_f}}$$

Therefore the variation of the Hamiltonian at the terminal time is

$$\frac{\partial Ham_0}{\partial x}(\tau_f) = P_{0_x}^T\frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + P_{0_x}^T\frac{\partial f}{\partial \tau_f}\frac{\partial \tau_f}{\partial x} + f^T(y,u,\tau_f)\left[\frac{\partial P_{0_x}}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial P_{0_x}}{\partial \tau_f}\frac{\partial \tau_f}{\partial x} \right]$$

$$= \Psi_{0_{\tau_f}}^T \frac{\partial \nu_0}{\partial x} - \left(\frac{\partial \phi_{0_{\tau_f}}}{\partial y} - \nu_0^T \frac{\partial \Psi_{0_{\tau_f}}}{\partial y} \right) \frac{\partial y}{\partial x} - \left(\frac{\partial \phi_{0_{\tau_f}}}{\partial \tau_f} - \nu_0^T \frac{\partial \Psi_{0_{\tau_f}}}{\partial \tau_f} \right) \frac{\partial \tau_f}{\partial x}$$

The variation in the parameters associated with the two-point boundary value problem with respect to variations in the initial states can be found from these sets of equations. Notice these equations are linear in the unknown parameters $\frac{\partial y}{\partial x}$, $\frac{\partial \tau_f}{\partial x}$, and $\frac{\partial \nu_0}{\partial x}$.

8.2 Expansion of the Euler-Lagrange Equations

This section attempts to relate the results of the expansion of the HJB-PDE to the results derived from expanding the ordinary differential Euler-Lagrange equations as was done in [19].

The states, control, and Lagrange multipliers are all expanded in an asymptotic series with expansion parameter ε. Thus,

$$y(\tau) = y_0(\tau) + \varepsilon y_1(\tau) + O(\varepsilon^2) \tag{8.6}$$

$$u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + O(\varepsilon^2) \tag{8.7}$$

$$\lambda_y(\tau) = \lambda_{0_y}(\tau) + \varepsilon \lambda_{1_y}(\tau) + O(\varepsilon^2) \tag{8.8}$$

These expansion equations are used in a Taylor series expansion of the dynamics:

$$f_0 = f(y, u, \tau) \tag{8.9}$$

$$f_1 = f_u(y, u, \tau) u_1 + f_y(y, u, \tau) y_1 \tag{8.10}$$

$$\vdots \qquad\qquad \vdots$$

$$g_0 = g(y, u, \tau) \tag{8.11}$$

$$g_1 \;=\; g_u(y, u, \tau) u_1 + g_y(y, u, \tau) y_1 \tag{8.12}$$

$$\vdots \qquad\qquad \vdots$$

where the notation is the same as previously presented, i.e., the state and dynamics y, f, and g are n-vectors and the control u and independent variable τ are scalars.

When expanded in the small parameter ε, the optimality condition

$$\lambda_y^T \left(\frac{\partial f(y, u, \tau)}{\partial u} + \varepsilon \frac{\partial g(y, u, \tau)}{\partial u} \right) = 0$$

becomes

$$\lambda_{0_y}^T \frac{\partial f(y, u, \tau)}{\partial u} \;=\; 0$$

$$\lambda_{1_y}^T \frac{\partial f(y, u, \tau)}{\partial u} + \lambda_{0_y}^T \left[\frac{\partial f_1(y, u, \tau)}{\partial u} + \frac{\partial g(y, u, \tau)}{\partial u} \right] \;=\; 0$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

after the coefficient terms of like powers of ε have been collected.

Thus the first-order expansion term for the optimal control is

$$u_1 = - \left[\lambda_{0_y}^T f_{uu}(y, u, \tau) \right]^{-1} \left[f_u^T(y, u, \tau) \lambda_{1_y} + \lambda_{0_y}^T f_{uy}(y, u, \tau) y_1 + \lambda_{0_y}^T g_u(y, u, \tau) \right]$$

This result is substituted into the Euler-Lagrange ordinary differential equations.

8.2.1 Expansion of the State Equations

The differential state equations

$$\frac{dy}{d\tau} = f(y, u, \tau) + \varepsilon g(y, u, \tau)$$

with initial conditons $y(\tau = t) = x$, can be expanded in the small parameter ε as

$$\ddot{y}_0 + \varepsilon \ddot{y}_1 + \varepsilon^2 \ddot{y}_2 + \ldots = f_0 + \varepsilon(f_1 + g_0) + \varepsilon^2(f_2 + g_1) + \ldots$$

Collecting like powers of ε leads to

$$\dot{y}_0 = f(y, u, \tau) \tag{8.13}$$

$$\dot{y}_1 = f_u(y, u, \tau) u_1 + f_y(y, u, \tau) y_1$$
$$+ g(y, u, \tau) \tag{8.14}$$

$$\vdots \qquad\qquad \vdots$$

The initial conditions are $y_0(t) = x_0$, $y_1(t) = x_1 = 0$.

Thus,

$$\dot{y}_0(\tau) = f(y, u, \tau)$$
$$\dot{y}_1(\tau) = \left\{ \left[f_y - f_u \left(\lambda_{0_y}^T f_{uu} \right)^{-1} \lambda_{0_y}^T f_{uy} \right] y_1 - f_u \left(\lambda_{0_y}^T f_{uu} \right)^{-1} f_u^T \lambda_{1_y} \right.$$
$$\left. + \left[g - f_u \left(\lambda_{0_y}^T f_{uu} \right)^{-1} \lambda_{0_y}^T g_u \right] \right\}$$

when the expansion terms of the control law are inserted into the state equations.

8.2.2 Expansion of the Lagrange Multiplier Equations

Expanding the differential equation for the Lagrange Multipliers

$$\dot{\lambda}_y = -(f_y^T + \varepsilon g_y^T) \lambda_y$$

produces the following set of equations

$$\dot{\lambda}_{0_y} = -f_y^T(y, u, \tau) \lambda_{0_y} \tag{8.15}$$

$$\dot{\lambda}_{1_y} = -f_y^T(y, u, \tau)\lambda_{1_y} - f_{1_y}^T\lambda_{0_y} - g_y^T(y, u, \tau)\lambda_{0_y} \qquad (8.16)$$

$$\vdots \qquad\qquad \vdots$$

Therefore,

$$\dot{\lambda}_{0_y} = -\frac{\partial f(y, u, \tau)^T}{\partial y}\lambda_{0_y} \qquad (8.17)$$

$$\dot{\lambda}_{1_y} = -\frac{\partial f(y, u, \tau)^T}{\partial y}\lambda_{1_y} - \left[\frac{\partial f_1^T(y, u, \tau)}{\partial y} + \frac{\partial g^T(y, u, \tau)}{\partial y}\right]\lambda_{0_y} \qquad (8.18)$$

Consequently, the differential equations which describe the first-order expansion terms are

$$\frac{d}{d\tau}\begin{bmatrix} y_1 \\ \lambda_{1_y} \end{bmatrix} = \begin{bmatrix} A_{yy} & A_{y\lambda_y} \\ A_{\lambda_y y} & A_{\lambda_y \lambda_y} \end{bmatrix}\begin{bmatrix} y_1 \\ \lambda_{1_y} \end{bmatrix}$$

where

$$A_{yy} = \left[f_y - f_u\left(\lambda_{0_y}^T f_{uu}\right)^{-1}\lambda_{0_y}^T f_{uy}\right] \qquad (8.19)$$

$$A_{y\lambda_y} = -f_u\left(\lambda_{0_y}^T f_{uu}\right)^{-1} f_u^T \qquad (8.20)$$

$$A_{\lambda_y y} = -\left[\lambda_{0_y}^T f_{yy} - \lambda_{0_y}^T f_{uy}\left(\lambda_{0_y}^T f_{uu}\right)^{-1}\lambda_{0_y}^T f_{yu}\right] \qquad (8.21)$$

$$A_{\lambda_y \lambda_y} = -\left[f_y^T - \lambda_{0_y}^T f_{uy}\left(\lambda_{0_y}^T f_{uu}\right)^{-1} f_u^T\right] \qquad (8.22)$$

The solution to this set of coupled differential equations is

$$\begin{bmatrix} y_1(\tau_{0_f}) \\ \lambda_{1_y}(\tau_{0_f}) \end{bmatrix} = \Phi_A(\tau_{0_f}, t)\begin{bmatrix} y_1(t) \\ \lambda_{1_y}(t) \end{bmatrix} + \int_t^{\tau_{0_f}} \Phi_A(\tau_{0_f}, \tau)G(\tau)d\tau$$

with $G(\tau)$ representing the forcing terms.

$$G(\tau) = \begin{bmatrix} g - f_u\left(\lambda_{0_y}^T f_{uu}\right)^{-1} g_u^T\lambda_{0_y} \\ -g_y^T\lambda_{0_y} + \lambda_{0_y}^T f_{uy}\left(\lambda_{0_y} f_{uu}\right)^{-1} g_u^T\lambda_{0_y} \end{bmatrix}$$

This is the same transition matrix as derived for the variations of the Optimal Return Function. Notice that the forcing terms are also the same as those derived for the variation of the Optimal Return Function.

8.3 Expansion of the Boundary Conditions

In order to expand the terminal boundary conditions properly the states must be expanded about the zeroth-order path and the zeroth-order final time. Thus,

$$\tau_f \;=\; \tau_{0_f} + \varepsilon\tau_{1_f} + O(\varepsilon^2) \tag{8.23}$$

$$y(\tau_f) \;=\; y_0(\tau_{0_f}) + \varepsilon\left[y_1(\tau_{0_f}) + \dot{y}_0(\tau_{0_f})\tau_{1_f}\right] + O(\varepsilon^2) \tag{8.24}$$

$$=\; y_{0_f} + \varepsilon y_{1_f} + O(\varepsilon^2) \tag{8.25}$$

Next expand the terminal boundary conditions with respect to the small parameter ε:

$$\Psi(y_f, \tau_f) \;=\; \sum_{i=0}^{\infty} \Psi_i(y_f, \tau_f)\varepsilon^i = 0$$

$$=\; \left[\Psi_0(y_f, \tau_f) + \varepsilon\Psi_1(y_f, \tau_f) + \varepsilon^2\Psi_2(y_f, \tau_f) + O(\varepsilon^3)\right]$$

The Taylor series expansion of the terminal boundary conditions is

$$\Psi(y_f, \tau_f) \;=\; \sum_{i=0}^{\infty}\left[\left(\frac{1}{i!}\frac{\partial^i \Psi(y_f^0, \tau_f^0)}{\partial y_f^i}\right)\left(\sum_{j=1}^{\infty} y_{j_f}\varepsilon^j\right)^i + \left(\frac{1}{i!}\frac{\partial^i \Psi(y_f^0, \tau_f^0)}{\partial \tau_f^i}\right)\left(\sum_{j=1}^{\infty} \tau_{j_f}\varepsilon^j\right)^i\right]$$

$$=\; \Psi(y_f, \tau_f)\Big|_{\varepsilon=0} + \varepsilon\frac{\partial\Psi}{\partial y_f}\Big|_{\varepsilon=0}y_{1_f} + \varepsilon\frac{\partial\Psi}{\partial \tau_f}\Big|_{\varepsilon=0}\tau_{1_f} + O(\varepsilon^2)$$

Collecting like terms in the expansion parameter ε results in

$$\Psi_0(y_f, \tau_f) \;=\; \Psi(y_f, \tau_f)\Big|_{\varepsilon=0} = 0 \tag{8.26}$$

$$\Psi_1(y_f, \tau_f) \;=\; \frac{\partial\Psi}{\partial y_f}\Big|_{\varepsilon=0}y_{1_f} + \frac{\partial\Psi}{\partial \tau_f}\Big|_{\varepsilon=0}\tau_{1_f} = 0 \tag{8.27}$$

$$\vdots \qquad\qquad \vdots$$

8.3.1 Expansion of the Transversality Conditions

Expanding the Hamiltonian $\lambda_y^T(f + \varepsilon g)$ in terms of the small parameter ε produces the following equations

$$
\begin{aligned}
Ham_0 &= \lambda_{0_y}^T f(y, u, \tau) \\
Ham_1 &= \lambda_{0_y}^T g(y, u, \tau) + \lambda_{1_y}^T f(y, u, \tau) + \lambda_{0_y}^T f_y(y, u, \tau) y_1 \qquad (8.28) \\
&\quad\vdots \qquad\qquad\qquad \vdots
\end{aligned}
$$

with boundary conditions at the terminal time

$$
\begin{aligned}
Ham_0(\tau_{0_f}) &= -\frac{\partial \phi(y_{0_f}, \tau_{0_f})}{\partial \tau_f} - \nu_0^T \frac{\partial \Psi(y_{0_f}, \tau_{0_f})}{\partial \tau_f} \\
Ham_1(\tau_f) &= -\frac{\partial \phi_{y_f}(y_{0_f}, \tau_{0_f})}{\partial \tau_f} y_{1_f} - \frac{\partial \phi_{\tau_f}(y_{0_f}, \tau_{0_f})}{\partial \tau_f} \tau_{1_f} - \nu_1^T \frac{\partial \Psi(y_{0_f}, \tau_{0_f})}{\partial \tau_f} \\
&\quad - \nu_0^T \frac{\partial \Psi_{y_f}(y_{0_f}, \tau_{0_f})}{\partial \tau_f} y_{1_f} - \nu_0^T \frac{\partial \Psi_{\tau_f}(y_{0_f}, \tau_{0_f})}{\partial \tau_f} \tau_{1_f}
\end{aligned}
$$

Equating the first-order expansion terms of these relationships, the equation for the first-order Hamiltonian at the terminal boundary becomes

$$
\begin{aligned}
&\lambda_{0_y}^T(\tau_{0_f}) g(y_{0_f}, u_{0_f}, \tau_{0_f}) + \left[\lambda_{1_y}^T(\tau_{0_f}) + \dot{\lambda}_{0_y}^T(\tau_{0_f}) \tau_{1_f} \right] f(y_{0_f}, u_{0_f}, \tau_{0_f}) \\
&+ \lambda_{0_y}^T(\tau_{0_f}) f_y(y_{0_f}, u_{0_f}, \tau_{0_f}) y_{1_f} = -\frac{\partial \phi_{y_f}(y_{0_f}, \tau_{0_f})}{\partial \tau_f} y_{1_f} - \frac{\partial \phi_{\tau_f}(y_{0_f}, \tau_{0_f})}{\partial \tau_f} \tau_{1_f} \\
&- \nu_0^T \frac{\partial \Psi_{y_f}(y_{0_f}, \tau_{0_f})}{\partial \tau_f} y_{1_f} - \nu_0^T \frac{\partial \Psi_{\tau_f}(y_{0_f}, \tau_{0_f})}{\partial \tau_f} \tau_{1_f} - \nu_1^T \frac{\partial \Psi(y_{0_f}, \tau_{0_f})}{\partial \tau_f}
\end{aligned}
$$

Now proceeding in a similar fashion, the terminal Lagrange Multipliers, determined by the relationship $\lambda_y^T(\tau_f) = \phi_{y_f}(y_f, \tau_f) + \nu^T \Psi_{y_f}(y_f, \tau_f)$, become

$$
\lambda_{1_y}(\tau_{0_f}) = -\frac{\partial \phi(y_{0_f}, \tau_{0_f})}{\partial y_f} - \nu_0^T \frac{\partial \Psi(y_{0_f}, \tau_{0_f})}{\partial y_f} \qquad (8.29)
$$

$$\dot{\lambda}_{0_y}(\tau_{0_f}) + \dot{\lambda}_{0_y}(\tau_{0_f})\tau_{1_f} = -\frac{\partial \phi_{y_f}(y_{0_f}, \tau_{0_f})}{\partial y_f}y_{1_f} - \frac{\partial \phi_{\tau_f}(y_{0_f}, \tau_{0_f})}{\partial y_f}\tau_{1_f}$$

$$-\nu_1^T \frac{\partial \Psi(y_{0_f}, \tau_{0_f})}{\partial y_f} \qquad -\nu_0^T \frac{\partial \Psi_{y_f}(y_{0_f}, \tau_{0_f})}{\partial y_f}y_{1_f} - \nu_0^T \frac{\partial \Psi_{\tau_f}(y_{0_f}, \tau_{0_f})}{\partial y_f}\tau_{1_f}$$

The expansion of the transversality conditions and the terminal boundary conditions produces the boundary conditions for the first-order two-point boundary value problem. As in the case of the variation of the Optimal Return Function, the unknown parameters, (y_{1_f}, ν_1, and τ_{1_f}), can be found by solving a set of linear algebraic equations.

8.4 Solution to the First-Order Problem

The solution to the first-order two-point boundary value problem is found by use of

$$\begin{bmatrix} y_1(\tau_{0_f}) \\ \lambda_{1_y}(\tau_{0_f}) \end{bmatrix} = \Phi_A(\tau_{0_f}, t)\begin{bmatrix} y_1(t) \\ \lambda_{1_y}(t) \end{bmatrix} + \int_t^{\tau_{0_f}} \Phi_A(\tau_{0_f}, \tau)G(\tau)d\tau \qquad (8.30)$$

subject to the terminal boundary conditions given for $y_1(\tau_{0_f})$ and $\lambda_{1_y}(\tau_{0_f})$ in Eqs. (8.27 - 8.30) and subject to the Hamiltonian transversality condition Eq. (8.29). Also recall that the initial states are considered known and are zeroth-order terms. Thus the first-order initial states are set equal to zero, $y_1(t) = 0$. The unknowns which need to be found are the initial and terminal first-order Lagrange multiplier terms, i.e. $\lambda_{1_y}(t)$ and ν_1, and the first-order term in the expansion of the final time τ_{1_f}.

For a trajectory that includes the staging condition, the terminal conditions remain the same but the form of the solution becomes

$$\begin{bmatrix} y_1(\tau_{0_f}) \\ \lambda_{1_y}(\tau_{0_f}) \end{bmatrix} = \Phi_{A_2}(\tau_{0_f}, \tau_{stage})\Phi_{A_1}(\tau_{stage}, t)\begin{bmatrix} y_1(t) \\ \lambda_{1_y}(t) \end{bmatrix}$$

$$+ \int_{\tau_{stage}}^{\tau_{0f}} \Phi_{A_2}(\tau_{0f}, \tau) G(\tau) d\tau \quad + \quad \Phi_{A_2}(\tau_{0f}, \tau_{stage}) \int_{t}^{\tau_{stage}} \Phi_{A_1}(\tau_{stage}, \tau) G(\tau) d\tau$$

where the state transition matrices Φ_{A_1} and Φ_{A_2} represent the state transition matrix over the first stage and second stage subarcs, respectively.

Notice that the form of the solution determined by expanding the Euler-Lagrange equations and the terminal boundary conditions in terms of the states, the control, and the Lagrange multipliers is equivalent to the solution found by the expansion of the Hamilton-Jacobi-Bellman partial differential equation. The forcing terms and the transition matrix used in the quadratures are the same. The first-order boundary conditions derived in this section are identical to the variation of the zeroth-order boundary conditions which are used in the HJB expansion method to determine the change in the parameters of the zeroth-order solution with respect to a change in the initial states. In the HJB expansion the variations $\frac{\partial y}{\partial x}$ and $\frac{\partial P_{0x}}{\partial x}$ are dependent on the changes in the constant parameters or the constants of the motion. Any admissible variation in the initial conditions must still generate a trajectory which satisfies the terminal conditions. Thus the variations in the boundary conditions with respect to changes in the initial states determine the change in the constants of the motion. And these changes are embedded in the solution of the variations in the state and Lagrange multipliers, $\frac{\partial y}{\partial x}$ and $\frac{\partial P_{0x}}{\partial x}$, which are used to generate the first-order correction terms. In contrast, the first-order boundary conditions derived from the Euler-Lagrange equations (which are equivalent to the variation of the zeroth-order boundary conditions) are explicitly used in solving the two-point boundary-value problem (8.30).

The similarity of the two techniques is not surprising since solving the

HJB-PDE is equivalent to solving the first-order Euler-Lagrange equations. The reason for this equivalency will be presented more clearly in the next section.

8.5 Solutions to First-Order Linear Partial Differential Equations

The solution to first-order partial differential equations is described in [11, 20]. In this section it will be shown that the canonical system of Euler-Lagrange differential equations is identical to the system of characteristic ordinary differential equations used to solve the partial differential equation. This is presented for the partial differential equation in two independent variables but the case of n-independent variables is a straightforward extension. First consider a partial differential equation of the form

$$F(x, t, P_x, P_t) = a(x,t)P_t + b(x,t)P_x - c(x,t) = 0 \qquad (8.31)$$

where a, b, and c are given functions and are considered continuous, as are their first derivatives, in the region of interest. The solution of this partial differential equation is called the integral surface and is denoted as $P(x,t)$. Since the coefficient terms (a, b, c) are not explicitly dependent on the solution P this is a linear rather than quasi-linear partial differential equation. The tangent plane to the integral surface P at the point $Q(x, t, P)$ is defined by the relationship (8.31) and the normal to the tangent plane is given by the directions P_x, P_t, and -1. The partial differential equation implies that the normal to the integral surface $< P_x, P_t, -1 >$ is perpendicular to a vector $< a, b, c >$ and so the vector $< a, b, c >$ must be tangent to any integral

surface at point Q. The vector $< a, b, c >$ is called the Monge vector. The tangent planes to all integral surfaces through the point $Q(x, t, P)$ belong to a family of planes which are described by the relation for the normal as

$$dt : dx : dP = a : b : c$$

The direction of the Monge vector at the point Q forms a characteristic line element ds. The directions of the Monge vectors form a directional field in the (x, t, P)-space. To solve the partial differential equation (8.31) the surfaces which fit the Monge vector must be found. Every surface whose tangent plane is tangent to the Monge vector at the point is a solution to the partial differential equation. The characteristic curves of the partial differential equation are the integral curves of the direction field and are defined by a set of ordinary differential equations. If the characteristic curves are considered a function of a parameter s then along the curves the characteristic equations become

$$\frac{dt}{ds} = a \qquad \frac{dx}{ds} = b \qquad \frac{dP}{ds} = c \qquad (8.32)$$

Thus a general solution surface can be generated independently of the initial data as a one-parameter family of characteristic curves.

For the initial value problem the manifold of possible integral surfaces can be created and the unique solution depends on the initial conditions of the problem, i.e. $y(s = 0) = x$, $\tau(s = 0) = t$, $P(x, t)$ =constant. Starting with a curve S in space the solution to the partial differential equation is sought. The curve S is projected onto the (x, t)-space and an integral solution $P(x, t)$ is to be found, see figure (8.5). Through each point of the space curve a family of characteristic curves can be generated according to the characteristic equations.

These curves form a surface and all characteristic curves lie within this integral surface.

The solution is sought at a point off of the initial data curve and in the direction of the characteristic curve. Thus the solution becomes a function of two variables, the initial state ($y(s = 0) = x$, $\tau(s = 0) = t$) and the running parameter s along the characteristic curve. As a consequence the solution to the integral surface $P(x, t)$ can be written as a function of s only with x, t replaced by their respective solutions along the characteristic direction and with fixed initial conditions at $s = 0$. This can be done if the characteristic solutions for x, t can be inverted to obtain functions dependent on $y(s = 0) = x, \tau(s = 0) = t$. The transformation between the two sets can only occur if the Jacobian is nonzero. In this case a unique solution exists to the initial value problem.

If the characteristic curve and the projection onto the x, t plane of the tangent to the curve S are identical then the curve S is a characteristic curve. Mathematically this happens if the Jacobian is equal to zero for every point along the curve. This is the relationship that was obtained in chapter 2 for the Hamilton-Jacobi-Bellman equation where $a = 1$ and $b = f(x, u, t)$. The solution obtained along the characteristic by definition must stay along the initial data curve. The implications of this result are that an infinite number of solutions exist for the integral surfaces which solve the partial differential equation and which pass through the curve S. Since the ordinary differential equations for the characteristic curves require the integration of a, b, and c which are known data along the the space curve S and since the projection of S in the x, t plane is the curve S itself, the integral solution $P(x, t)$ can be found by integration of $\frac{dP}{ds} = c$. The unique solution to the problem is determined
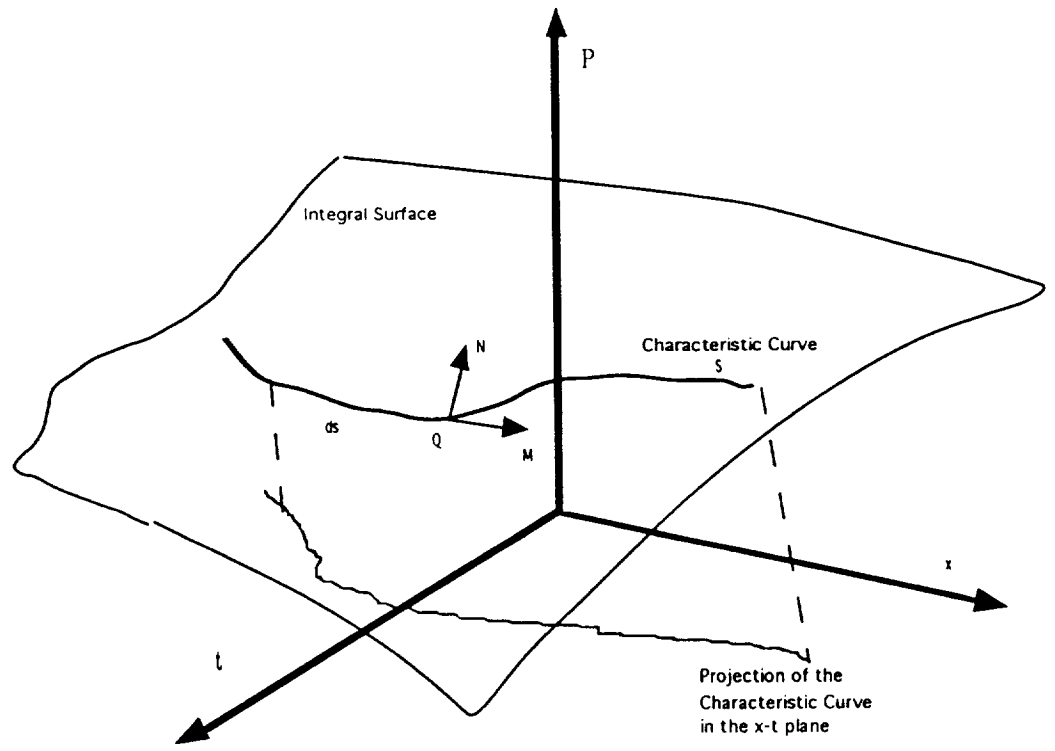
Figure 8.1: Geometric Interpretation of Integral Surface

by the initial conditions of the ordinary differential equations and the terminal boundary conditions such that P can not be specified arbitrarily at the initial point $y(s = 0) = x$, $\tau(s = 0) = t$.

In [10] the relationship to the calculus of variations approach is simply derived. If a particular trajectory is the optimal trajectory then the Hamilton-Jacobi-Bellman equation must hold at each point and thus can be written with respect to the states y and independent variable τ. The fundamental equation is

$$P_y^T(y, \tau)\left(f(y, u, \tau) + \varepsilon g(y, u, \tau)\right) + P_\tau(y, \tau) = 0$$

and the optimal return function depends only on the cost at the terminal manifold. Therefore, the optimal return function is constant and the total time derivative must be zero at each point in the path. The partial derivative with respect to y is

$$
\begin{aligned}
0 \;=\; & \left(f^T(y, u, \tau) + \varepsilon g^T(y, u, \tau)\right)\frac{\partial P_y}{\partial y} + \frac{\partial P_\tau}{\partial y} + P_y^T\left(\frac{\partial f(y, u, \tau)}{\partial y} + \varepsilon\frac{\partial g(y, u, \tau)}{\partial y}\right) \\
& + P_y^T\left(\frac{\partial f(y, u, \tau)}{\partial u}\frac{\partial u}{\partial y} + \varepsilon\frac{\partial g(y, u, \tau)}{\partial u}\frac{\partial u}{\partial y}\right)
\end{aligned}
$$

By the chain rule for differentiation

$$\frac{dP_y}{d\tau} = \left(f^T(y, u, \tau) + \varepsilon g^T(y, u, \tau)\right)\frac{\partial P_y}{\partial y} + \frac{\partial P_\tau}{\partial y}$$

Thus a system of ordinary differential equations is obtained which is evaluated along an optimal trajectory and is satisfied by the partial derivatives of the Optimal Return Function $P(y, \tau)$.

$$
\begin{aligned}
\frac{dP_y}{d\tau} \;=\; & -P_y^T\left(\frac{\partial f(y, u, \tau)}{\partial y} + \varepsilon\frac{\partial g(y, u, \tau)}{\partial y}\right) \\
& -P_y^T\left(\frac{\partial f(y, u, \tau)}{\partial u}\frac{\partial u}{\partial y} + \varepsilon\frac{\partial g(y, u, \tau)}{\partial u}\frac{\partial u}{\partial y}\right)
\end{aligned}
\tag{8.33}
$$

Now the optimality condition is

$$P_y^T \left(\frac{\partial f(y, u, \tau)}{\partial u} + \varepsilon \frac{\partial g(y, u, \tau)}{\partial u} \right) = 0$$

Therefore,

$$\frac{dP_y}{d\tau} = -P_y^T \left(\frac{\partial f(y, u, \tau)}{\partial y} + \varepsilon \frac{\partial g(y, u, \tau)}{\partial y} \right) \tag{8.34}$$

which is the familiar Lagrange multiplier rule for the optimal trajectory.

The material presented is intended to clarify the relationship between the solution to the first-order, linear, partial differential equation which is the Hamilton-Jacobi-Bellman equation and the solution to the Euler-Lagrange equations. The point to remember is that the solution to the partial differential equation is given by characteristics generated by ordinary differential equations which are the equivalent to the canonical Euler-Lagrange equations.

8.6 Formulation of First-Order Correction Terms for the ALS Problem

In this section the solution for the ALS problem using the new perturbation method , i.e. the perturbed Euler-Lagrange equations, is presented.

The state transition matrix is determined by integrating

$$\frac{d}{d\tau} \Phi(\tau, t) = \begin{bmatrix} A_{yy} & A_{y\lambda_y} \\ A_{\lambda_y y} & A_{\lambda_y \lambda_y} \end{bmatrix} \Phi(\tau, t), \qquad \Phi(t, t) = I$$

where the A matrix was presented in Eq. (8.22)

For the ALS problem the primary and perturbation dynamics are

$$f(y, u, \tau) = \begin{bmatrix} \dot{w} \\ \dot{u} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} g_s - \frac{T_{vac}}{m} \sin \theta_p \\ \frac{T_{vac}}{m} \cos \theta_p \\ -w \end{bmatrix}$$

and

$$g(y, u, \tau) = \begin{bmatrix} \frac{r_e}{h_s}\left(\frac{A_x}{m} - \frac{u^2}{(r_e+h)} - g_s\frac{h(2r_e+h)}{(r_e+h)^2} + \frac{n}{m}pA_e\sin\theta_p\right) \\ \frac{r_e}{h_s}\left(\frac{A_x}{m} + \frac{uw}{(r_e+h)} - \frac{n}{m}pA_e\cos\theta_p\right) \\ 0 \end{bmatrix}$$

where A_x and A_z represent the aerodynamic forces in the x- and z-directions, respectively. Also n is the number of engines per stage and p and A_e are the pressure and the engine exit area. Remember that the thrust was modelled as $T = T_{vac} - npA_e$.

The first-order partial derivatives of the primary dynamics are

$$f_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad f_u = \begin{bmatrix} -\frac{T_{vac}}{m}\cos\theta_p \\ -\frac{T_{vac}}{m}\sin\theta_p \\ 0 \end{bmatrix}$$

and all the second-order derivatives are identically zero except f_{uu},

$$f_{uy} = f_{yu} = f_{yy} = 0 \qquad f_{uu} = \begin{bmatrix} \frac{T_{vac}}{m}\sin\theta_p \\ -\frac{T_{vac}}{m}\cos\theta_p \\ 0 \end{bmatrix}$$

Therefore the matrix A in the differential equation defining the state transition matrix becomes

$$A = \begin{bmatrix} 0 & 0 & 0 & a_{14} & a_{15} & 0 \\ 0 & 0 & 0 & a_{24} & a_{25} & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with

$$a_{14} = \frac{T_{vac}}{m}\frac{\cos^2\theta_p}{(\lambda_u\cos\theta_p - \lambda_w\sin\theta_p)} \qquad a_{25} = \frac{T_{vac}}{m}\frac{\sin^2\theta_p}{(\lambda_u\cos\theta_p - \lambda_w\sin\theta_p)}$$

$$a_{15} = a_{24} = \frac{T_{vac}}{m}\frac{\cos\theta_p\sin\theta_p}{(\lambda_u\cos\theta_p - \lambda_w\sin\theta_p)}$$

An analytic form for the state transition matrix can thus be obtained.

$$\Phi(\tau,t) = \begin{bmatrix} 1 & 0 & 0 & \Phi_{14} & \Phi_{15} & \Phi_{16} \\ 0 & 1 & 0 & \Phi_{24} & \Phi_{25} & \Phi_{26} \\ -(\tau-t) & 0 & 1 & \Phi_{34} & \Phi_{35} & \Phi_{36} \\ 0 & 0 & 0 & 1 & 0 & (\tau-t) \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\Phi_{14} = -\frac{C_u^2}{\sigma a^{3/2}}\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$
$$-\frac{2C_u^2}{\sigma a\Delta}\left[\frac{(bcm-2ac+b^2)}{\sqrt{cm^2+bm+a}} - \frac{(bcm_i-2ac+b^2)}{\sqrt{cm_i^2+bm_i+a}}\right]$$

$$\Phi_{15} = \Phi_{24} = \frac{C_u\overline{C}_w}{\sigma \dot{m}a^{3/2}}\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$
$$+\frac{2C_u\overline{C}_w}{\sigma \dot{m}a\Delta}\left[\frac{(bcm-2ac+b^2)}{\sqrt{cm^2+bm+a}} - \frac{(bcm_i-2ac+b^2)}{\sqrt{cm_i^2+bm_i+a}}\right]$$
$$+\frac{2C_u\lambda_h}{\sigma \dot{m}\Delta}\left[\frac{(2cm+b)}{\sqrt{cm^2+bm+a}} - \frac{(2cm_i+b)}{\sqrt{cm_i^2+bm_i+a}}\right]$$

$$\Phi_{16} = -\frac{C_u^2 m_i}{\sigma \dot{m}a^{3/2}}\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$
$$-\frac{2C_u^2 m_i}{\sigma \dot{m}a\Delta}\left[\frac{(bcm-2ac+b^2)}{\sqrt{cm^2+bm+a}} - \frac{(bcm_i-2ac+b^2)}{\sqrt{cm_i^2+bm_i+a}}\right]$$
$$-\frac{2C_u^2}{\sigma \dot{m}\Delta}\left[\frac{(2cm+b)}{\sqrt{cm^2+bm+a}} - \frac{(2cm_i+b)}{\sqrt{cm_i^2+bm_i+a}}\right]$$

$$\Phi_{25} = -\frac{\overline{C}_w^2}{\sigma a^{3/2}}\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right) - \sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$
$$-\frac{2\overline{C}_w^2}{\sigma a\Delta}\left[\frac{(bcm-2ac+b^2)}{\sqrt{cm^2+bm+a}} - \frac{(bcm_i-2ac+b^2)}{\sqrt{cm_i^2+bm_i+a}}\right]$$
$$+\frac{2b}{\sigma\Delta}\left[\frac{(2cm+b)}{\sqrt{cm^2+bm+a}} - \frac{(2cm_i+b)}{\sqrt{cm_i^2+bm_i+a}}\right]$$

$$-\frac{2c}{\sigma\Delta}\left[\frac{(2a+bm)}{\sqrt{cm^2+bm+a}}-\frac{(2a+bm_i)}{\sqrt{cm_i^2+bm_i+a}}\right]$$

$$\Phi_{26} = \frac{C_u\overline{C}_w m_i}{\sigma\dot{m}a^{3/2}}\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$

$$+\frac{2C_u\overline{C}_w m_i}{\sigma\dot{m}a\Delta}\left[\frac{(bcm-2ac+b^2)}{\sqrt{cm^2+bm+a}}-\frac{(bcm_i-2ac+b^2)}{\sqrt{cm_i^2+bm_i+a}}\right]$$

$$+\frac{2C_u}{\sigma\dot{m}\Delta}\left(\overline{C}_w+\frac{\lambda_h}{\dot{m}}m_i\right)\left[\frac{(2cm+b)}{\sqrt{cm^2+bm+a}}-\frac{(2cm_i+b)}{\sqrt{cm_i^2+bm_i+a}}\right]$$

$$+\frac{2C_u\lambda_h}{\sigma\dot{m}^2\Delta}\left[\frac{(2a+bm)}{\sqrt{cm^2+bm+a}}-\frac{(2a+bm_i)}{\sqrt{cm_i^2+bm_i+a}}\right]$$

$$\Phi_{34} = -\frac{C_u^2}{\sigma\dot{m}a^{3/2}}m\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$

$$-\frac{2C_u^2 b}{\sigma\dot{m}a\Delta}\left[\sqrt{cm^2+bm+a}-\sqrt{cm_i^2+bm_i+a}\right]$$

$$+\frac{2C_u^2}{\sigma\dot{m}a\Delta}(m-m_i)\frac{(bcm_i-2ac+b^2)}{\sqrt{cm_i^2+bm_i+a}}$$

$$\Phi_{35} = \frac{C_u\overline{C}_w}{\sigma\dot{m}a^{3/2}}m\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$

$$+\frac{2C_u}{\sigma\dot{m}\Delta}\left(\frac{2\lambda_h}{\dot{m}}+\frac{b\overline{C}_w}{a}\right)\left[\sqrt{cm^2+bm+a}-\sqrt{cm_i^2+bm_i+a}\right]$$

$$-\frac{2C_u}{\sigma\dot{m}\Delta}\left[\frac{\overline{C}_w}{a}\left(bcm_i-2ac+b^2\right)+\frac{\lambda_h}{\dot{m}}(2cm_i+b)\right]\frac{(m-m_i)}{\sqrt{cm_i^2+bm_i+a}}$$

$$\Phi_{36} = -\frac{C_u^2 m_i}{\sigma\dot{m}^2 a^{3/2}}m\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$

$$-\frac{2C_u^2}{\sigma\dot{m}^2\Delta}\left(2+\frac{bm_i}{a}\right)\left[\sqrt{cm^2+bm+a}-\sqrt{cm_i^2+bm_i+a}\right]$$

$$+\frac{2C_u^2}{\sigma\dot{m}^2\Delta}\left[\frac{m_i}{a}\left(bcm_i-2ac+b^2\right)+(2cm_i+b)\right]\frac{(m-m_i)}{\sqrt{cm_i^2+bm_i+a}}$$

The first-order partial derivatives of the perturbation dynamics are

$$
g_y = \begin{bmatrix} \frac{r_e}{mh_s}\frac{\partial A_z}{\partial w} & \frac{r_e}{h_s}\left(\frac{1}{m}\frac{\partial A_z}{\partial u} - \frac{2u}{(r_e+h)}\right) & \frac{\partial \dot{w}}{\partial h} \\ \frac{r_e}{h_s}\left(\frac{1}{m}\frac{\partial A_x}{\partial w} + \frac{w}{(r_e+h)}\right) & \frac{r_e}{h_s}\left(\frac{1}{m}\frac{\partial A_x}{\partial u} + \frac{w}{(r_e+h)}\right) & \frac{\partial \dot{u}}{\partial h} \\ 0 & 0 & 0 \end{bmatrix}
$$

and

$$
g_{\theta_p} = \frac{r_e}{mh_s}\begin{bmatrix} \left(\frac{\partial A_z}{\partial \theta_p} + npA_e\cos\theta_p\right) \\ \left(\frac{\partial A_x}{\partial \theta_p} + npA_e\sin\theta_p\right) \\ 0 \end{bmatrix}
$$

where

$$
\frac{\partial \dot{w}}{\partial h} = \frac{r_e}{h_s}\left(\frac{1}{m}\frac{\partial A_z}{\partial h} + \frac{u^2}{(r_e+h)^2} - \frac{2g_s r_e^2}{(r_e+h)^3} - \frac{npA_e}{mh_p}\sin\theta_p\right) \quad (8.35)
$$

$$
\frac{\partial \dot{u}}{\partial h} = \frac{r_e}{h_s}\left(\frac{1}{m}\frac{\partial A_x}{\partial h} - \frac{uw}{(r_e+h)^2} + \frac{npA_e}{mh_p}\cos\theta_p\right) \quad (8.36)
$$

The forcing terms in the quadratures pertaining to the first-order correction terms can now be calculated. Recall

$$
G(\tau) = \begin{bmatrix} g - f_u\left(\lambda_{0_y}^T f_{uu}\right)^{-1}g_u^T\lambda_{0_y} \\ -g_y^T\lambda_{0_y} + \lambda_{0_y}^T f_{uy}\left(\lambda_{0_y}f_{uu}\right)^{-1}g_u^T\lambda_{0_y} \end{bmatrix}
$$

which becomes

$$
G(\tau) = \begin{bmatrix} g - f_u\left(\lambda_{0_y}^T f_{uu}\right)^{-1}g_u^T\lambda_{0_y} \\ -g_y^T\lambda_{0_y} \end{bmatrix}
$$

The aerodynamic forces are considered positive in the x- and z-directions respectively, and are

$$
A_x = -0.5\rho S(u^2 + w^2)\left[C_D(M,\alpha)\cos\gamma + C_L(M,\alpha)\sin\gamma\right] \quad (8.37)
$$

$$
A_z = 0.5\rho S(u^2 + w^2)\left[C_D(M,\alpha)\sin\gamma - C_L(M,\alpha)\cos\gamma\right] \quad (8.38)
$$

The angle-of-attack is a function of the pitch angle and the flight path angle, $\alpha = \theta_p - \gamma$. The atmospheric density and pressure are modelled as exponentials, i.e. $\rho = \rho_s exp(-h/h_s)$ and $p = p_s exp(-h/h_p)$. The mach number is a

function of velocity and altitude with $M = (u^2 + w^2)/sos$. The speed-of-sound is calculated by $sos = \sqrt{\Gamma \frac{p}{\rho}}$, where Γ is the specific heat ratio for air and is assigned a value of 1.4 . Lastly, the flight path angle is represented in the Cartesian coordinate system as $\tan \gamma = -\frac{w}{u}$. From these relationships all the partial derivatives needed to calculate the forcing terms $G(\tau)$ can be determined.

8.7 Results

The solution to the launch problem was first attempted for initial conditions associated with staging. At these altitudes the aerodynamic forces are small enough that they may correctly be consider perturbation terms. The results of the new peturbation method (expansion of the Euler-Lagrange equations) show excellent agreement with the optimal solution. Note that the entire first-order correction is available since this method is valid as an open loop solution as is shown in figure (8.2). The results also agree exactly at the initial point of the path with the previous results using the old method (HJB). Table (8.1) lists the relevent values.

Next, the solution for initial conditions at a time of 35 seconds was sought. Once again the solution via the new method matched exactly the result obtained using the old method. To obtain agreement with the optimal trajectory, the aerodynamic pulse functions were utilized in the same manner as previously discussed. Consequently, the first-order solution closely approximated the optimal solution. The values at the initial point are also included in Table (8.1) and plots of the open loop profiles are in figure (8.3). The solution in a feedback configuration is presented in figures (8.4-8.6). Presented are the

Method	$T_0 = 35.$		$T_0 = 153.54$	
	P_v	P_γ	P_v	P_γ
new first	-2.2535	-843.12	-0.8547	-908.35
new pulse	-1.3925	-153.50	-	-
HJB first	-2.2535	-843.12	-0.8547	-908.36
HJB pulse	-1.3954	-153.82	-	-
shooting	-1.2752	-139.48	-0.8151	-860.63

Table 8.1: Comparison of open loop results

Method	final time (sec.)	final weight (lbs.)	B.C. error	
			γ deg	h ft
new first	369.91	329295.	0.0026	.219
new pulse	369.59	330578.	.0014	-.144
HJB first	369.91	329293.	.03	-.002
HJB pulse	369.59	330576.	.0001	.0007
shooting	369.57	330678.	-	-

Table 8.2: Comparison of closed loop results

profiles of the flight path angle Lagrange multipliers, the velocity Lagrange multipliers, and the control over the entire trajectory. In the closed loop solution the values are practically identical with a slight difference in the accuracy upon which the terminal conditions are met. In order to be consistent with the results chapter, the second stage aerodynamic model was used throughout the flight. Table (8.2) verifies that the results are the same for the two perturbation methods.

To summarize, the expansion of the Hamilton-Jacobi-Bellman equation is equivalent to the expansion of the Euler-Lagrange canonical equations with respect to the states, control, and the Lagrange multipliers. The theoretical and geometric concepts behind the solution of first-order partial differential equations were also discussed. The reason for the equivalency of the two methods is that the result obtained by solving the partial differential equation and differentiating with respect to the initial states is identical to the result obtained by the solution of the ordinary differential equations that represent the characteristic equations for the partial differential equation. For the Hamilton-Jacobi-Bellman equation the characteristic equations are the Euler-Lagrange equations. As expected, the solutions obtained using the two perturbation techniques are identical. While the calculus of variations approach took a longer amount of computation time, the entire open loop trajectory can be generated. Because of this fact the update to the feedback solution need not be computed as often and thus the overall computational time can be reduced. At the expense of this speed comes the additional burden of integrating a state transition matrix rather than calculating the partial derivatives of the zeroth-order solution. While the state transition matrix approach is easier to understand
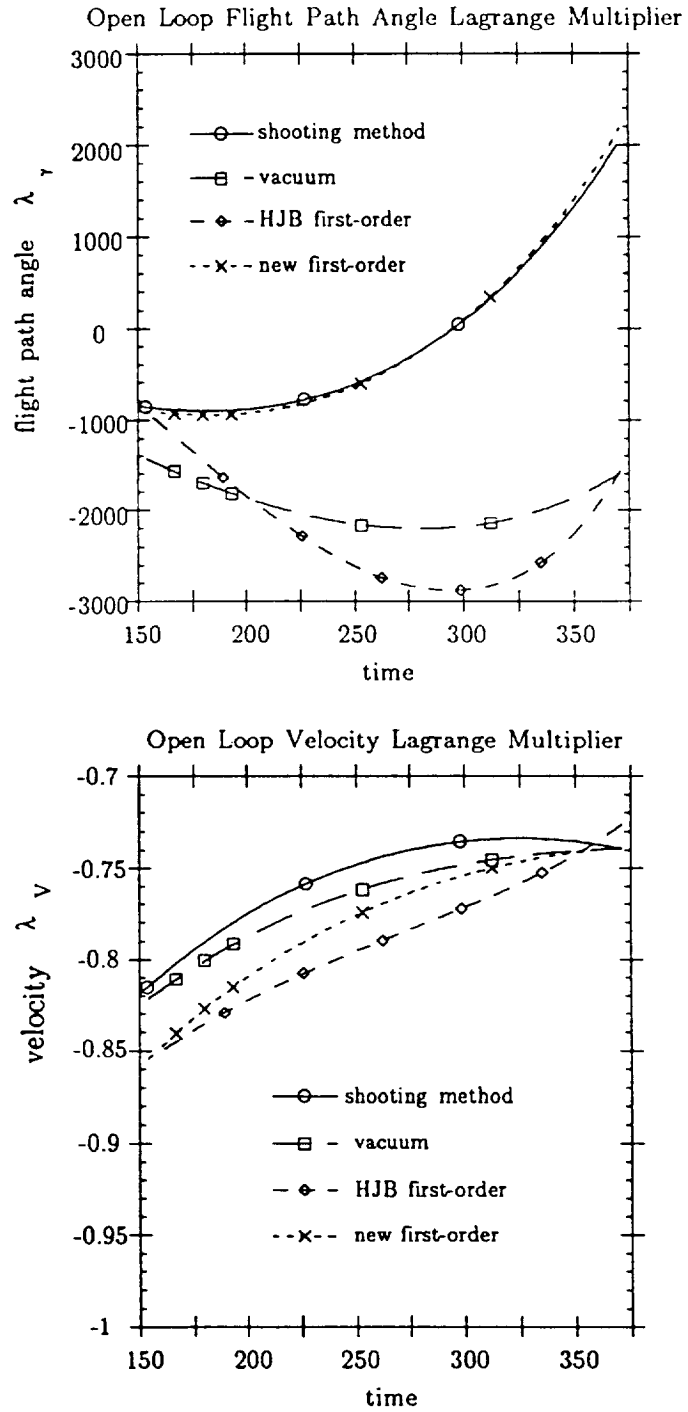
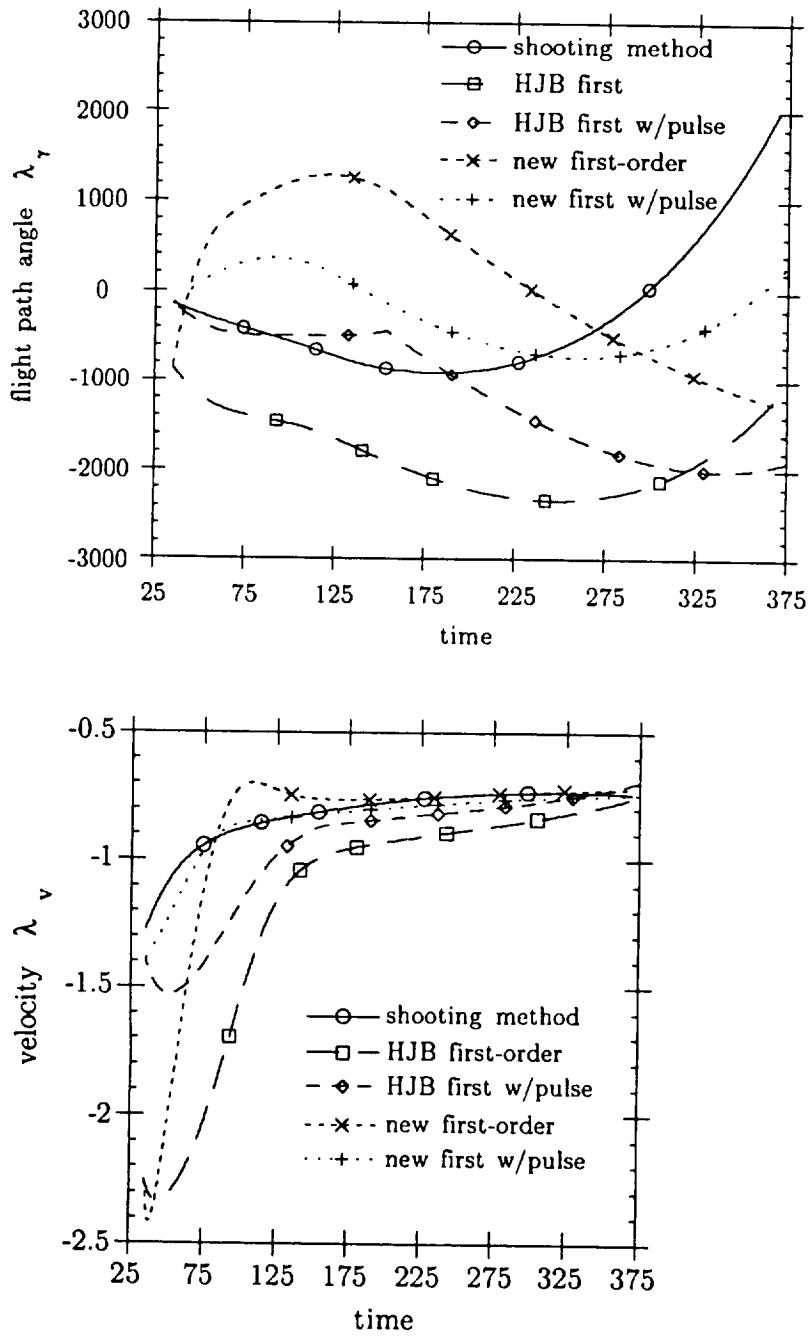Figure 8.2: Open loop solution for Lagrange multipliers at staging conditions

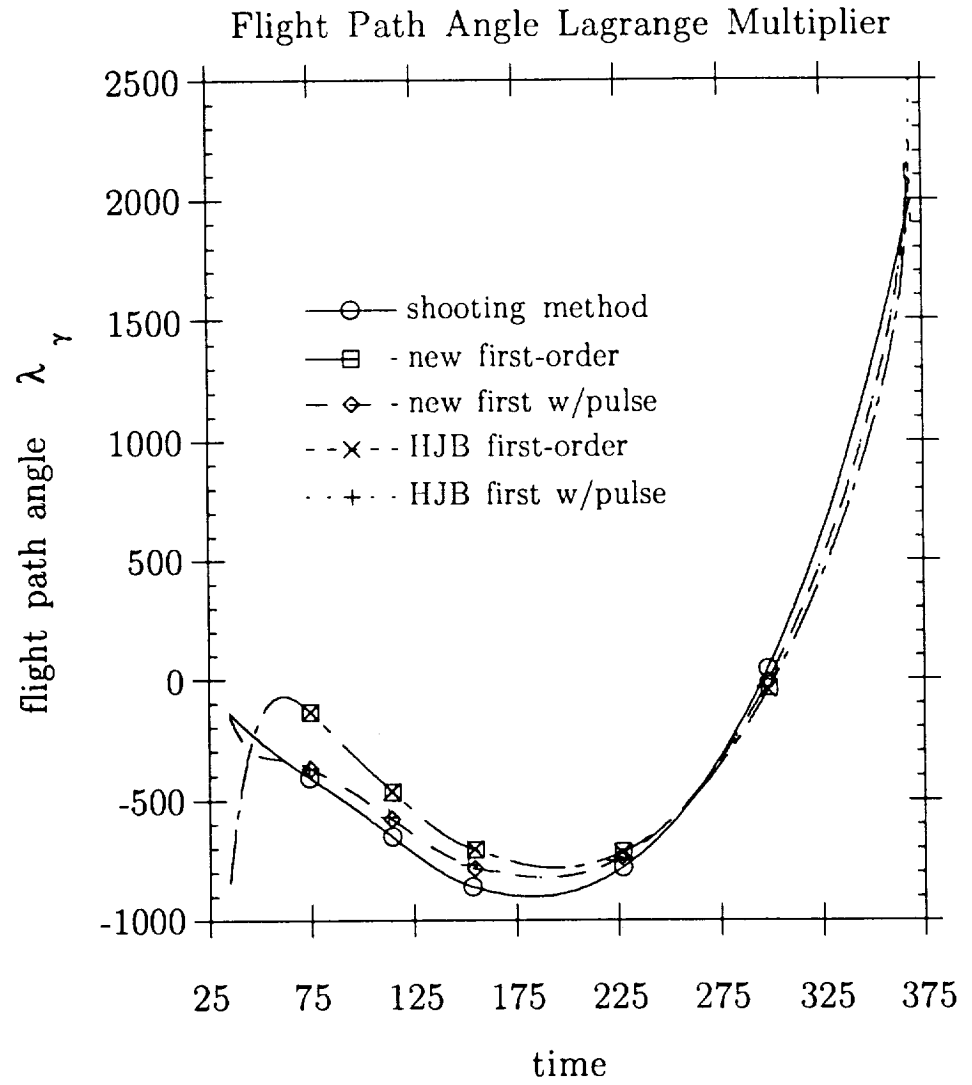Figure 8.3: Open loop solution for Lagrange multipliers at first stage initial conditions

Figure 8.4: Closed loop solution for flight path angle Lagrange multipliers
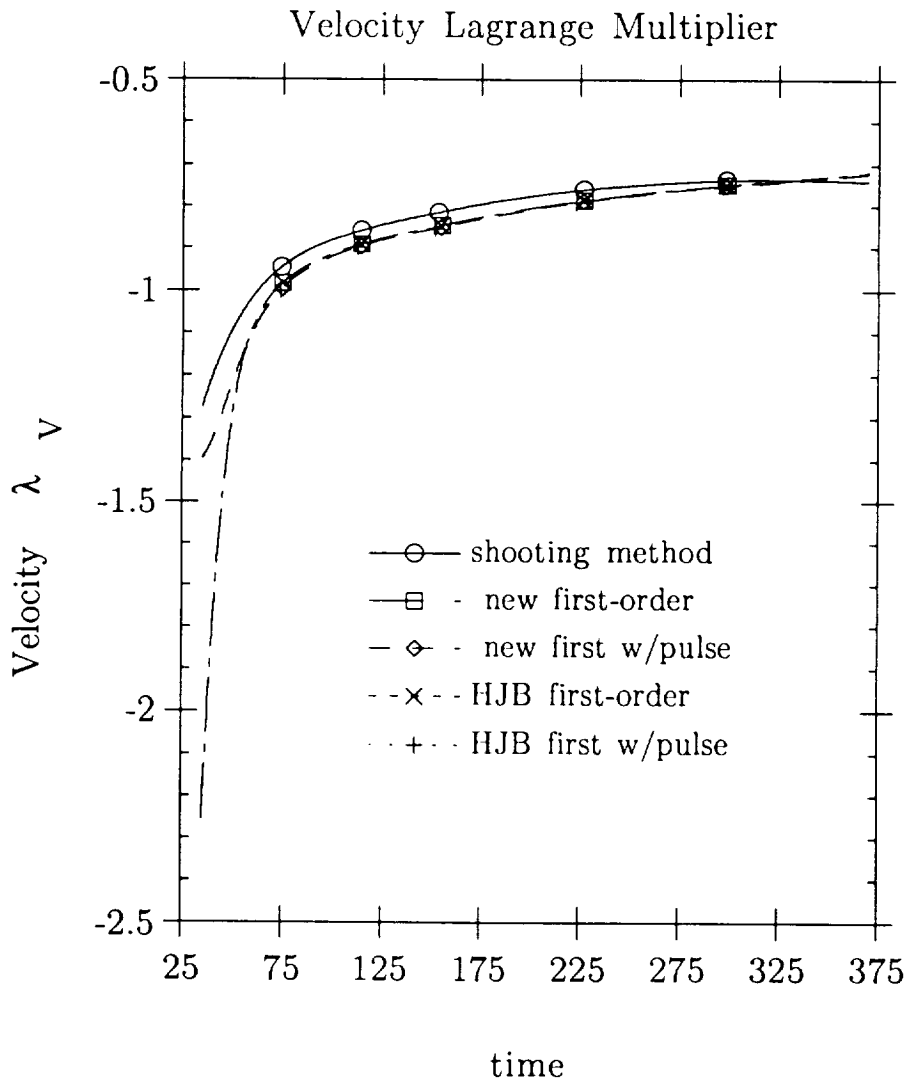
Figure 8.5: Closed loop solution for velocity Lagrange multipliers
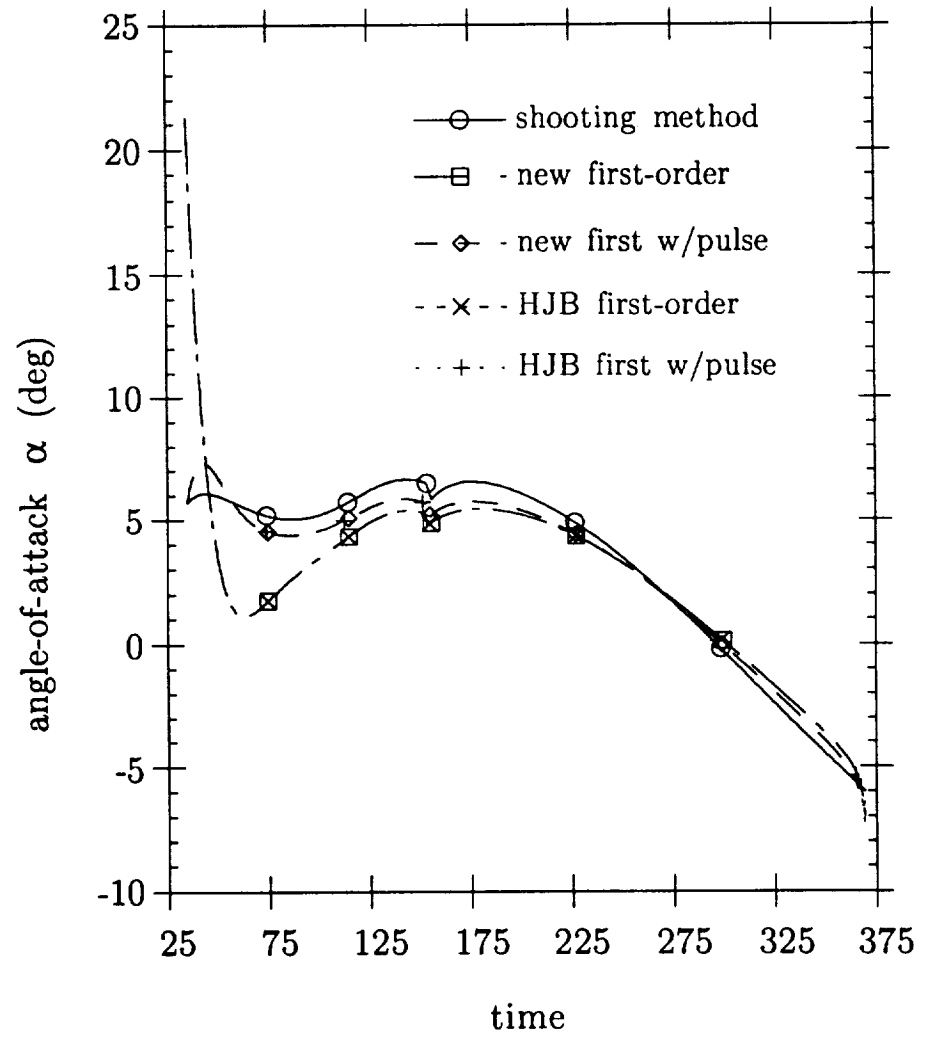
Figure 8.6: Closed loop solution for angle-of-attack

than the embedded nature of the HJB solution, the state transition matrix can be more difficult to obtain than the corresponding partial derivatives needed by the HJB expansion method. Also note that the Hamilton-Jacobi-Bellman equation can be written as a stochastic equation and can thus handle random disturbances.

Chapter 9

Conclusions

The technique for applying the expansion of the Hamilton-Jacobi-Bellman partial differential equation to derive a real-time guidance scheme has been presented. The problem of launching a vehicle into orbit was simulated for flight restricted to the equatorial plane. Difficulties arose at low altitudes due to the large and highly nonlinear aerodynamic forces. While the expansion method gave reasonable results, the use of aerodynamic pulse functions in order to reshape the zeroth-order trajectory was vital to matching the optimal trajectory. Thus it is essential that the zeroth-order path, upon which the higher-order corrections are based, resembles the optimal solution such that the assumed perturbing effects are indeed small. Based on the difficulties caused by the aerodynamics and the modelling of these aerodynamics a few suggestions are offered. First, the modelling of atmospheric and aerodynamic terms should be adequate well beyond the domain of the optimal solution. This is especially necessary if in some manner the zeroth-order trajectory significantly deviates from the optimal trajectory. It is also suggested that this technique would work better with a symmetric version of the ALS vehicle configuration by eliminating the irregular behavior of the Hamiltonian. The results of this research showed that the idea of using perturbation theory to perform real-time on-line guidance is a valid one. The improvement in computational speed and effort over the generation of optimal solutions is evident. Still, the technique as

presented here was not designed for computational efficiency. To that end, use of parallel processing in the integration of the quadratures is proposed. This has the potential of decreasing the computational time even further. Lastly, one of the goals for deriving an on-line guidance scheme is to provide abort opportunities or instantaneous changes in the terminal destination. It should be remembered that this method always provides a nominal path which satisfies the terminal constraints while an improvement in the performance is obtained by the first-order correction terms. Because of the robustness of the solution to the zeroth-order analytic two-point boundary problem, in-flight aborts can be easily included.

More sophisticated modelling, such as including an oblate Earth model and wind profiles, can easily be done since these effects can be considered perturbations and included in the problem in the higher-order correction terms. The result would be to integrate some additional quadratures. The technique can also be extended in a straightforward manner to flight in three-dimensions in order to reach a point in space. See appendix[A] for the zeroth-order analytic solution. This is done through the addition of the out-of-plane equations but with an accompanying increase in the complexity of the problem. It is expected that the three-dimensional solution will increase the payload available at orbital insertion due to the benefits of the rotational effects of the Earth. The inclusion of a dynamic pressure point inequality constraint is also feasible. In other ALS studies [4, 12] it has been shown that since the rocket cannot be throttled, the vehicle only touches the dynamic pressure constraint at a point. It is suggested here that this point inequality constraint can be included in the analytic solution of the zeroth-order problem as an interior point constraint

[see appendix C]. This new zeroth-order trajectory would avoid the large aerodynamic correction terms found in the unconstrained optimization problem. These large correction terms are due to the aerodynamic forces encountered when flying through the region of high dynamic pressure. The solution to a zeroth-order trajectory including a dynamic pressure constraint represents the most important step in the evolution of this method for implementation as a real-time, on-line guidance scheme for the launch problem.

Research is in order, on the inclusion of variable state and control inequality constraints in the expansion of the Hamilton-Jacobi-Bellman equation, so as to generalize the class of optimization problems amenable to expansion techniques. Future studies are required on the general properties of the validity of the asymptotic expansion of the HJB equation. For example, the question of whether or not the asymptotic expansion is uniformly convergent remains to be answered. This is especially true in light of the use of the ad hoc aerodynamic pulse functions. Also, the approximate optimal guidance scheme must be made robust with respect to parameter variations and stochastic disturbances in order to be implemented as a real-time on-line guidance scheme. Possible solution methods [21, 22, 23] have been proposed to handle the more realistic situation of a nondeterministic environment. The use of the Hamilton-Jacobi-Bellman equation is particular advantageous under these circumstances since a stochastic version of the equation exists. With the zeroth-order trajectory providing full state information, the best solution in the presence of random disturbances should be obtainable.

Appendix A

Zeroth-Order Solution for Three-Dimensional Flight

The analytic zeroth-order solution is derived once again by a transformation of coordinate system. A canonical transformation from the wind axis to the rectangular or local horizon coordinate frame allows the zeroth-order problem to be solved analytically. The solution is in closed form up to some constants that can be determined numerically. By making the transformation

$$u = V \cos \gamma \cos \chi \tag{A.1}$$

$$v = V \cos \gamma \sin \chi \tag{A.2}$$

$$w = -V \sin \gamma \tag{A.3}$$

the zeroth-order equations of motion in a cartesian coordinate frame become

$$\dot{X} = u \tag{A.4}$$

$$\dot{Y} = v \tag{A.5}$$

$$\dot{h} = -w \tag{A.6}$$

$$\dot{u} = \frac{T}{m} \cos \theta_p \cos \psi \tag{A.7}$$

$$\dot{v} = \frac{T}{m} \cos \theta_p \sin \psi \tag{A.8}$$

$$\dot{w} = -\frac{T}{m} \sin \theta_p + g_s \tag{A.9}$$

$$\dot{m} = -\sigma T$$

where the Thrust pitch attitude, $\theta_p = \alpha + \gamma$, and the Thrust yaw angle, $\psi = \chi + \beta$, become the control variables for the zeroth-order solution.

The optimization problem to zeroth-order is solved by the Hamiltonian

$$H = \lambda_X u + \lambda_Y v - \lambda_h w + \lambda_u \frac{T}{m} \cos\theta_p \cos\psi +$$
$$\lambda_v \frac{T}{m} \cos\theta_p \sin\psi + \lambda_w(-\frac{T}{m} \sin\theta_p + g_s) \qquad (A.10)$$

The zeroth-order control laws determined by the optimality conditons are:

$$H_\psi = \frac{T}{m} \cos\theta_p (\lambda_v \cos\psi - \lambda_u \sin\psi) = 0 \qquad (A.11)$$

$$H_{\theta_p} = -(\lambda_u \cos\psi + \lambda_v \sin\psi)\frac{T}{m} \sin\theta_p - \frac{T}{m}\lambda_w \cos\theta_p = 0 \qquad (A.12)$$

Thus,

$$\tan\psi = \frac{\lambda_v}{\lambda_u}$$
$$\cos\psi = -\frac{\lambda_u}{\sqrt{\lambda_u^2 + \lambda_v^2}}$$
$$\sin\psi = -\frac{\lambda_v}{\sqrt{\lambda_u^2 + \lambda_v^2}} \qquad (A.13)$$

and using these forms for the cosine and sine of ψ, $\quad \theta_p$ is

$$\tan\theta_p = \frac{\lambda_w}{\sqrt{\lambda_u^2 + \lambda_v^2}}$$
$$\cos\theta_p = \frac{\sqrt{\lambda_u^2 + \lambda_v^2}}{\sqrt{\lambda_u^2 + \lambda_u^2 + \lambda_w^2}}$$
$$\sin\theta_p = \frac{\lambda_w}{\sqrt{\lambda_u^2 + \lambda_u^2 + \lambda_w^2}} \qquad (A.14)$$

Propagation of the Lagrange multipliers by $\dot{\lambda}_x = -H_x^T$ gives

$$\dot{\lambda}_X = 0$$
$$\dot{\lambda}_Y = 0$$
$$\dot{\lambda}_h = 0$$
$$\dot{\lambda}_u = -\lambda_X$$
$$\dot{\lambda}_v = -\lambda_Y$$
$$\dot{\lambda}_w = \lambda_h \tag{A.15}$$

with boundary conditions given by $\lambda_f = G_{xf}$

$$\lambda_X(\tau_f) = \nu_X, \quad \lambda_Y(\tau_f) = \nu_Y, \quad \lambda_h(\tau_f) = \nu_h, \quad \lambda_u(\tau_f) = \nu_u,$$
$$\lambda_v(\tau_f) = \nu_v, \quad \lambda_w(\tau_f) = \nu_w$$

where $\nu_X, \nu_Y, \nu_h, \nu_u, \nu_v, \nu_w$ are unknown Lagrange multipliers associated with the terminal constraints. The solutions to the adjoint differential equations are

$$\lambda_X = \nu_X$$
$$\lambda_Y = \nu_Y$$
$$\lambda_h = \nu_h$$
$$\lambda_u = C_u - \lambda_X(\tau - \tau_0)$$
$$\lambda_v = C_v - \lambda_Y(\tau - \tau_0)$$
$$\lambda_w = C_w + \lambda_h(\tau - \tau_0) \tag{A.16}$$

The equations of motion can be integrated by changing the independent variable from time to mass and using the mass equation to substitute mass for τ.

Therefore, the Lagrange multipliers become

$$\lambda_u = \overline{C}_u + \lambda_X \frac{m}{\sigma T} \tag{A.17}$$

$$\lambda_v = \overline{C}_v + \lambda_Y \frac{m}{\sigma T} \tag{A.18}$$

$$\lambda_w = \overline{C}_w - \lambda_h \frac{m}{\sigma T} \tag{A.19}$$

$$\lambda_u^2 + \lambda_v^2 + \lambda_w^2 = cm^2 + bm + a \tag{A.20}$$

where

$$c = \frac{\lambda_X^2 + \lambda_Y^2 + \lambda_h^2}{(\sigma T)^2} \tag{A.21}$$

$$b = \frac{2}{\sigma T}(\lambda_X \overline{C}_u + \lambda_Y \overline{C}_v - \lambda_h \overline{C}_w) \tag{A.22}$$

$$a = \overline{C}_u^2 + \overline{C}_v^2 + \overline{C}_w^2 \tag{A.23}$$

$$\overline{C}_u = C_u - \lambda_X \frac{m_0}{\sigma T} \tag{A.24}$$

$$\overline{C}_v = C_v - \lambda_Y \frac{m_0}{\sigma T} \tag{A.25}$$

$$\overline{C}_w = C_w + \lambda_h \frac{m_0}{\sigma T} \tag{A.26}$$

and the state equations become

$$\frac{du}{dm} = \frac{\lambda_u}{\sigma m \sqrt{cm^2 + bm + a}} \tag{A.27}$$

$$\frac{dv}{dm} = \frac{\lambda_v}{\sigma m \sqrt{cm^2 + bm + a}} \tag{A.28}$$

$$\frac{dw}{dm} = \frac{\lambda_w}{\sigma m \sqrt{cm^2 + bm + a}} - \frac{g_s}{\sigma T} \tag{A.29}$$

$$\frac{dX}{dm} = -\frac{u}{\sigma T} \tag{A.30}$$

$$\frac{dY}{dm} = -\frac{v}{\sigma T} \tag{A.31}$$

$$\frac{dh}{dm} = \frac{w}{\sigma T} \tag{A.32}$$

Note that $c > 0$, $a > 0$ and $\Delta \triangleq 4ac - b^2 > 0$ since

$$\Delta = \frac{4}{(\sigma T)^2} \left[(\lambda_X C_v - \lambda_Y C_u)^2 + (\lambda_X C_w + \lambda_h C_u)^2 + (\lambda_Y C_w + \lambda_h C_v)^2 \right] \quad \text{(A.33)}$$

Therefore, the integrations can be obtained from standard integrals. After some manipulation, solutions for the states are obtained in terms of the unknown constants as

$$
\begin{aligned}
u = \ & u_0 + \frac{\lambda_X}{\sigma^2 T \sqrt{c}} \left[\sinh^{-1}\left(\frac{2cm + b}{\sqrt{\Delta}} \right) - \sinh^{-1}\left(\frac{2cm_0 + b}{\sqrt{\Delta}} \right) \right] \\
& - \frac{\overline{C_u}}{\sigma \sqrt{a}} \left[\sinh^{-1}\left(\frac{2a + bm}{m\sqrt{\Delta}} \right) - \sinh^{-1}\left(\frac{2a + bm_0}{m_0\sqrt{\Delta}} \right) \right]
\end{aligned}
\quad \text{(A.34)}
$$

$$
\begin{aligned}
v = \ & v_0 + \frac{\lambda_Y}{\sigma^2 T \sqrt{c}} \left[\sinh^{-1}\left(\frac{2cm + b}{\sqrt{\Delta}} \right) - \sinh^{-1}\left(\frac{2cm_0 + b}{\sqrt{\Delta}} \right) \right] \\
& - \frac{\overline{C_v}}{\sigma \sqrt{a}} \left[\sinh^{-1}\left(\frac{2a + bm}{m\sqrt{\Delta}} \right) - \sinh^{-1}\left(\frac{2a + bm_0}{m_0\sqrt{\Delta}} \right) \right]
\end{aligned}
\quad \text{(A.35)}
$$

$$
\begin{aligned}
w = \ & w_0 - \frac{g_s}{\sigma T}(m - m_0) \\
& - \frac{\lambda_h}{\sigma^2 T \sqrt{c}} \left[\sinh^{-1}\left(\frac{2cm + b}{\sqrt{\Delta}} \right) - \sinh^{-1}\left(\frac{2cm_0 + b}{\sqrt{\Delta}} \right) \right] \\
& - \frac{\overline{C_w}}{\sigma \sqrt{a}} \left[\sinh^{-1}\left(\frac{2a + bm}{m\sqrt{\Delta}} \right) - \sinh^{-1}\left(\frac{2a + bm_0}{m_0\sqrt{\Delta}} \right) \right]
\end{aligned}
\quad \text{(A.36)}
$$

$$
\begin{aligned}
X = \ & X_0 - \frac{(m - m_0)}{\sigma T} u_0 \\
& - \frac{\lambda_X}{\sigma(\sigma T)^2 \sqrt{c}} (m + \frac{b}{2c}) \left[\sinh^{-1}\left(\frac{2cm + b}{\sqrt{\Delta}} \right) - \sinh^{-1}\left(\frac{2cm_0 + b}{\sqrt{\Delta}} \right) \right] \\
& + m\frac{\overline{C_u}}{\sigma(\sigma T) \sqrt{a}} \left[\sinh^{-1}\left(\frac{2a + bm}{m\sqrt{\Delta}} \right) - \sinh^{-1}\left(\frac{2a + bm_0}{m_0\sqrt{\Delta}} \right) \right] \\
& + \frac{\overline{C_u}}{\sigma(\sigma T) \sqrt{c}} \left[\sinh^{-1}\left(\frac{2cm + b}{\sqrt{\Delta}} \right) - \sinh^{-1}\left(\frac{2cm_0 + b}{\sqrt{\Delta}} \right) \right] \\
& - \frac{\lambda_X}{\sigma(\sigma T)^2 c} \left[\sqrt{cm_0^2 + bm_0 + a} - \sqrt{cm^2 + bm + a} \right]
\end{aligned}
\quad \text{(A.37)}
$$

$$
Y = \ Y_0 - \frac{(m - m_0)}{\sigma T} v_0
$$

$$-\frac{\lambda_Y}{\sigma(\sigma T)^2\sqrt{c}}\left(m+\frac{b}{2c}\right)\left[\sinh^{-1}\left(\frac{2cm+b}{\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2cm_0+b}{\sqrt{\Delta}}\right)\right]$$

$$+m\frac{\overline{C}_v}{\sigma(\sigma T)\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$

$$+\frac{\overline{C}_v}{\sigma(\sigma T)\sqrt{c}}\left[\sinh^{-1}\left(\frac{2cm+b}{\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2cm_0+b}{\sqrt{\Delta}}\right)\right]$$

$$-\frac{\lambda_Y}{\sigma(\sigma T)^2 c}\left[\sqrt{cm_0^2+bm_0+a}-\sqrt{cm^2+bm+a}\right] \tag{A.38}$$

$$h = h_0-\frac{g_s}{2(\sigma T)^2}(m-m_0)^2+\frac{(m-m_0)}{\sigma T}w_0$$

$$-\frac{\lambda_h}{\sigma(\sigma T)^2\sqrt{c}}\left(m+\frac{b}{2c}\right)\left[\sinh^{-1}\left(\frac{2cm+b}{\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2cm_0+b}{\sqrt{\Delta}}\right)\right]$$

$$-m\frac{\overline{C}_w}{\sigma(\sigma T)\sqrt{a}}\left[\sinh^{-1}\left(\frac{2a+bm}{m\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2a+bm_0}{m_0\sqrt{\Delta}}\right)\right]$$

$$-\frac{\overline{C}_w}{\sigma(\sigma T)\sqrt{c}}\left[\sinh^{-1}\left(\frac{2cm+b}{\sqrt{\Delta}}\right)-\sinh^{-1}\left(\frac{2cm_0+b}{\sqrt{\Delta}}\right)\right]$$

$$-\frac{\lambda_h}{\sigma(\sigma T)^2 c}\left[\sqrt{cm_0^2+bm_0+a}-\sqrt{cm^2+bm+a}\right] \tag{A.39}$$

There are seven unknown constants that are to be determined $m_f, C_u, C_v, C_w, \lambda_X, \lambda_Y, \lambda_h$. These unknown constants can be found using the initial and final states which are known, the six state equations above, and the transversality condition for the Hamiltonian.

These equations are valid for arcs before and after staging occurs. To determine a point on the trajectory after staging, given initial conditions before staging, the Weierstrass-Erdmann corner conditions can be used to relate the Lagrange multipliers before and after staging and thus link the two subarcs. Since the states are assumed continuous across the stage time and the change in mass is a known fixed quantity, all the Lagrange multipliers are continuous in time. The Return Function is thus continuous and constant across the stage

time. But since the analytic state equations are derived using mass as the independent variable, the equations for the states and Lagrange multipliers change across the stage. The form of the equations is the same but the initial values for the states in the equations are now replaced by the values of the states at staging before the discontinuity in mass. The mass flow rate, σT, will change and the initial time associated with the Lagrange multipliers becomes the stage time. Therefore,

$$
\begin{aligned}
\lambda_u(t) &= \left(\lambda_u(t_{st}) - \lambda_X \frac{m_{st2}}{\sigma T_{st2}}\right) + \lambda_X \frac{m}{\sigma T_{st2}} \\
&= \overline{C}_u + \lambda_X \frac{m}{\sigma T_{st2}} \\
\lambda_v(t) &= \left(\lambda_v(t_{st}) - \lambda_Y \frac{m_{st2}}{\sigma T_{st2}}\right) + \lambda_Y \frac{m}{\sigma T_{st2}} \\
&= \overline{C}_v + \lambda_Y \frac{m}{\sigma T_{st2}} \\
\lambda_w(t) &= \left(\lambda_w(t_{st}) + \lambda_h \frac{m_{st2}}{\sigma T_{st2}}\right) - \lambda_h \frac{m}{\sigma T_{st2}} \\
&= \overline{C}_w - \lambda_h \frac{m}{\sigma T_{st2}}
\end{aligned}
$$

Thus the constants a,b,and c in the analytic solution have the same form but the term $\overline{C}$ after staging becomes related to $\overline{C}$ before staging.

$$
\begin{aligned}
\overline{C}_u &= \overline{C}_u - \lambda_X \left(\frac{m_{st2}}{\sigma T_{st2}} - \frac{m_{st1}}{\sigma T_{st1}}\right) \\
\overline{C}_v &= \overline{C}_v - \lambda_Y \left(\frac{m_{st2}}{\sigma T_{st2}} - \frac{m_{st1}}{\sigma T_{st1}}\right) \\
\overline{C}_w &= \overline{C}_w + \lambda_h \left(\frac{m_{st2}}{\sigma T_{st2}} - \frac{m_{st1}}{\sigma T_{st1}}\right)
\end{aligned}
$$

Through a coordinate transformation back into the wind axis the angle of attack can be determined. Higher-order terms in the Lagrange multipliers can be found from expansion of the dynamic programming equation. The Optimal Return Function can be determined by integrating the perturbation terms

along the zeroth-order trajectory while taking into account variations in the final time due to changes in the initial states. Taking the partial derivative of the Return Function with respect to the initial states determines the Lagrange multipliers. Expanding to first-order the control and Lagrange multipliers in the control law determines the first-order approximation for the controls.

A.1 Zeroth-Order Coordinate Transformation

The analytic solution for the zeroth-order problem has been found in the Cartesian coordinate system but the equations of motion of the full system which includes the aerodynamic forces are written in the wind axes system. To derive the zeroth-order control and the first-order correction to the controls the transformation of coordinates and especially the transformation of the Lagrange multipliers must be known. The rotation from the wind axes to the local horizon frame is done by a canonical transformation.

$$
\begin{aligned}
u &= V \cos \gamma \cos \chi \\
v &= V \cos \gamma \sin \chi \\
w &= -V \sin \gamma
\end{aligned}
\tag{A.40}
$$

A necessary and sufficient condition for the transformation to be canonical is that the Hamiltonians be equivalent.

$$
\begin{aligned}
H_{LH} &= \lambda_X dX + \lambda_Y dY + \lambda_h dh + \lambda_u du + \lambda_v dv + \lambda_w dw \\
H_{Wind} &= \lambda_\theta d\theta + \lambda_\phi d\phi + \lambda_h dh + \lambda_V dV + \lambda_\gamma d\gamma + \lambda_\chi d\chi
\end{aligned}
$$

For the two reference frames, h and m are the same. Also, for the model $X = r_e\theta$ and $Y = r_e\phi$. Thus,

$$\lambda_\theta = r_e\lambda_X, \quad \lambda_\phi = r_e\lambda_Y, \quad \text{and} \quad \lambda_h \text{ is the same}$$

In order to equate the Hamiltonians, the Jacobian of the transformation from the wind axes to the local horizon axes is computed.

$$\lambda_u du + \lambda_v dv + \lambda_w dw = \lambda_V dV + \lambda_\gamma d\gamma + \lambda_\chi d\chi$$

So,

$$\begin{bmatrix} \lambda_V \\ \lambda_\gamma \\ \lambda_\chi \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial V} & \frac{\partial v}{\partial V} & \frac{\partial w}{\partial V} \\ \frac{\partial u}{\partial \gamma} & \frac{\partial v}{\partial \gamma} & \frac{\partial w}{\partial \gamma} \\ \frac{\partial u}{\partial \chi} & \frac{\partial v}{\partial \chi} & \frac{\partial w}{\partial \chi} \end{bmatrix} \begin{bmatrix} \lambda_u \\ \lambda_v \\ \lambda_w \end{bmatrix}$$

and thus

$$\begin{bmatrix} \lambda_V \\ \lambda_\gamma \\ \lambda_\chi \end{bmatrix} = \begin{bmatrix} \cos\gamma\cos\chi & \cos\gamma\sin\chi & -\sin\gamma \\ -V\sin\gamma\cos\chi & -V\sin\gamma\sin\chi & -V\cos\gamma \\ -V\cos\gamma\sin\chi & V\cos\gamma\cos\chi & 0 \end{bmatrix} \begin{bmatrix} \lambda_u \\ \lambda_v \\ \lambda_w \end{bmatrix}$$

Therefore,

$$\lambda_V = \lambda_u\cos\gamma\cos\chi + \lambda_v\cos\gamma\sin\chi - \lambda_w\sin\gamma \tag{A.41}$$

$$\lambda_\gamma = -V(\lambda_u\sin\gamma\cos\chi + \lambda_v\sin\gamma\sin\chi + \lambda_w\cos\gamma) \tag{A.42}$$

$$\lambda_\chi = V\cos\gamma(-\lambda_u\sin\chi + \lambda_v\cos\chi) \tag{A.43}$$
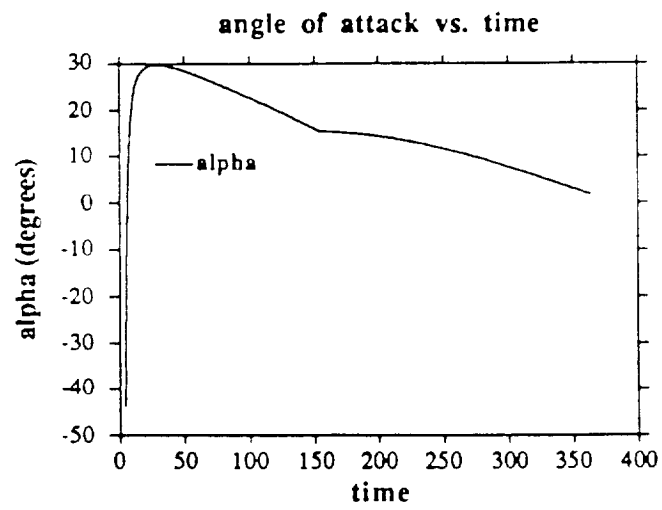
$$\lambda_\theta = r_e\lambda_X$$

$$\lambda_\phi = r_e\lambda_Y$$

and

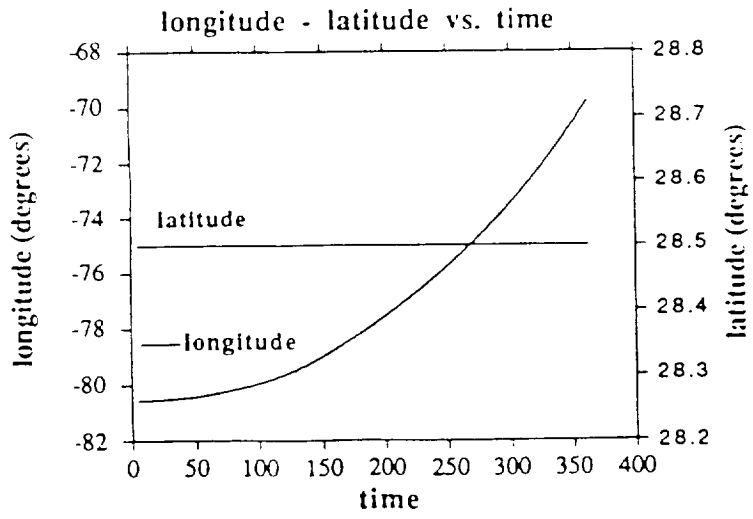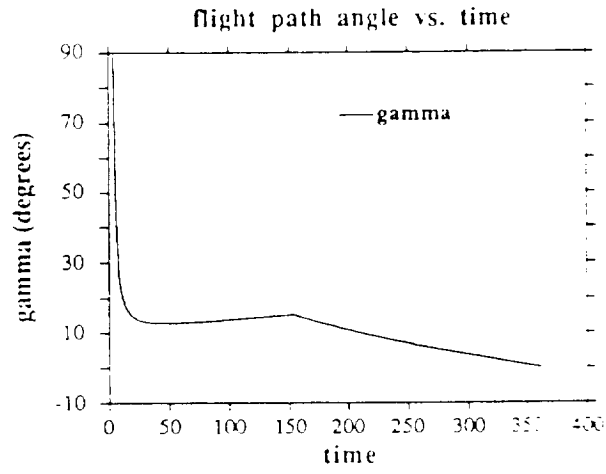$$V = \sqrt{u^2 + v^2 + w^2}$$

$$\tan\chi = \frac{v}{u}$$

$$\sin\gamma = -\frac{w}{V}$$

Plots are presented to show the characteristics of the solution for flight in a vacuum using the open loop analytic solution.

angle of attack vs. time

flight path angle vs. time

longitude - latitude vs. time

velocity vs. time

altitude vs. time
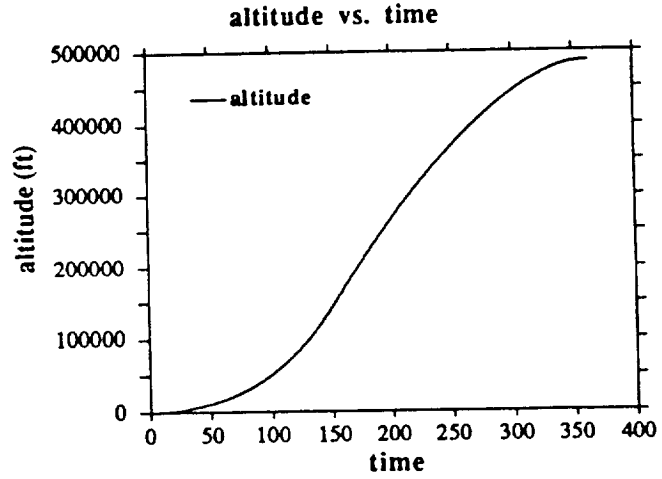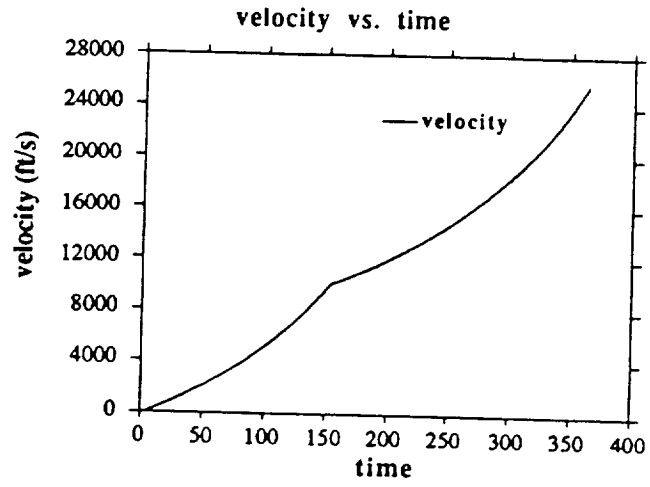
Appendix B

Canonical Transformations

The use of the canonical transformation of section 4.2 is essential in finding the closed-form solution to the zeroth-order problem because canonical transformations preserve the Hamiltonian form of the equations of motion in the new set of variables. A more thorough discussion of canonical transformations than what is presented here is in [24]. To transform between the generalized coordinates and generalized momenta or Lagrange multipliers (q, p) of one system to new variables (Q, P) of a new system, a set of transformation equations linking the two systems must be known. This link between the two systems can be derived from the generating function $S(q, Q, t)$.

$$\frac{d}{dt} S(q, Q, t) = L(q, \dot{q}, t) - \bar{L}(Q, \dot{Q}, t) \tag{B.1}$$

where $L = T - V$ is the Lagrangian of the respective system. Let the transformation equations be of the form

$$Q = Q(q, p, t) \qquad P = P(q, p, t) \tag{B.2}$$

with Hamiltonians associated with each set of variables such that the Hamiltonian equations are satisfied, i.e.,

$$H(q, p, t) = \sum p_i \dot{q}_i - L(q, \dot{q}, t) \tag{B.3}$$

$$p_i = \frac{\partial L}{\partial \dot{q}} \tag{B.4}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{B.5}$$

$$K(Q, P, t) = \sum P_i \dot{Q}_i - L(Q, \dot{Q}, t) \tag{B.6}$$

$$P_i = \frac{\partial L}{\partial \dot{Q}} \tag{B.7}$$

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \tag{B.8}$$

By solving Eqs. (B.3)-(B.6) for the Lagrangians and substituting into Eq. (B.1), the difference between two differential forms can be obtained as an exact differential.

$$\sum p_i dq_i - H dt - \sum P_i dQ_i + K dt = dS \tag{B.9}$$

This is the sufficient condition for a canonical transformation between the old variables (q, p) and the new variables (Q, P).

The generating function can be written as a total differential of the form

$$dS = \sum \frac{\partial S}{\partial q_i} dq_i + \sum \frac{\partial S}{\partial Q_i} dQ_i + \frac{\partial S}{\partial t} dt \tag{B.10}$$

Equating like terms of the differential dS yields

$$p_i = \frac{\partial S}{\partial q_i}, \quad P_i = \frac{\partial S}{\partial Q_i}, \quad K = H + \frac{\partial S}{\partial t} \tag{B.11}$$

A simplified form of these equations can be obtained. Assume that time is not changed in the transformation from one system to the other system. Therefore, t is an independent parameter and the value of dt is set to zero. Also, define a new function equal in value to the generating function but expressed in a different form

$$\psi(q, p, t) = S(q, Q, t) \tag{B.12}$$

Therefore, the variation of Eq. (B.1) is rewritten as

$$\sum p_i \delta q_i - \sum P_i \delta Q_i = \delta \psi \qquad (\text{B.13})$$

The equations

$$\frac{\partial \psi}{\partial q_i} = p_i - \sum_j P_j \frac{\partial Q_j}{\partial q_i} \qquad (\text{B.14})$$

$$\frac{\partial \psi}{\partial p_i} = -\sum_j P_j \frac{\partial Q_j}{\partial p_i} \qquad (\text{B.15})$$

are obtained by expressing the variations $\delta \psi$ and $P_i \delta Q_i$ in terms of the old variables. The new Hamiltonian is determined by integrating the expressions presented above to obtain ψ, and then calculating

$$K = H + \frac{\partial \psi}{\partial t} + \sum_i P_i \frac{\partial Q_i}{\partial t} \qquad (\text{B.16})$$

For a homogeneous canonical transformation the generating function S or ψ is identically zero and thus $d\psi$ is an exact differential and equal to zero. For a point canonical transformation as used in section 4.2, the transformation equations $Q_i = Q_i(q)$ are functions only of the generalized coordinates of the old system. The functions $Q_i(q)$ represent a full set of independent functions, therefore, the old variables q can be expressed in terms of the new variables Q using these same functions. This implies that the Jacobian determinant is not zero,

$$\left| \frac{\partial Q_i}{\partial q_j} \right| = \frac{\partial(Q_1, Q_2, ..., Q_n)}{\partial(q_1, q_2, ..., q_n)} \neq 0 \qquad (\text{B.17})$$

Since the transformation equations (Q, ψ) do not contain time, the Hamiltonians of the two systems are equal ($K = H$). This result is obtained by the use

of Eq. (B.16). The relationships expressed in Eqs. (B.14)-(B.15) become

$$0 \; = \; p_i - \sum_j P_j \frac{\partial Q_j}{\partial q_i} \tag{B.18}$$

$$0 \; = \; - \sum_j P_j \frac{\partial Q_j}{\partial p_i} \tag{B.19}$$

The Jacobian determinant of the new variables Q in terms of the old set of Lagrange multipliers p must be zero for a set of P's not all identically zero and is also required since the point transformation $Q = Q(q)$ was independent of the Lagrange multipliers p. The matrix $\frac{\partial Q}{\partial q}$ is used to determine the relationship (Eq. (B.18)) between the Lagrange multipliers of the two systems. The transformation between the wind axis coordinates and the Cartesian coordinates of section 4.2 is a canonical transformation as can be verified by the use of the canonical transformation equation (Eq. (B.13)).

Appendix C

Point Inequality Constraints

The inclusion of the a point inequality constraint on the dynamic pressure is discussed in this section. Due to structural load limits imposed on the ALS vehicle, the optimization problem must include a dynamic pressure inequality constraint. For the unconstrained optimization problem that is presented in this research, the correction terms to the zeroth-order solution become too large near the region of maximum dynamic pressure. For a rocket incapable of throttling, the optimal trajectory will not include a subarc on the boundary of the dynamic pressure constraint but instead will only touch the constraint at a point [4, 12]. This result would seem to indicate that the dynamic pressure inequality constraint can be handled in the same manner as an interior point constraint. Therefore, the Hamilton-Jacobi-Bellman equation can be split into subarcs before and after satisfying the interior point constraint. Let the optimal return function be

$$P(x, t) = \sum_{i=0}^{\infty} \epsilon^i P_i(x, t) \tag{C.1}$$

The point interior equality constraint is

$$N(y(t_1)) = 0 \tag{C.2}$$

where the constraint is a function of the states y at the time t_1 and the states are assumed continuous at the constraint. The system dynamics are defined

by

$$\ddot{y} = f(y, u, t) \tag{C.3}$$

and therefore the Hamiltonian is $H = P_x f$. Across the equality constraint

$$P_x(y, t_1-) = P_x(y, t_1+) + \Pi N_x(y, t_1) \tag{C.4}$$

$$P_x(y, t_1-)f(y, u, t_1-) = P_x(y, t_1+)f(y, u, t_1+) \tag{C.5}$$

where Π is the Lagrange multiplier used to adjoin the constraint to the performance index. These equations are the corner conditions derived using the calculus of variations [9]. From the solution of the Hamilton-Jacobi-Bellman first-order partial differential equation, the higher-order terms of the expansion of the optimal return function are

$$P_i(x, t) = - \int_t^{t_1-} R_i d\tau - \int_{t_1+}^{t_f} R_i d\tau \qquad i = 1, 2, \ldots \tag{C.6}$$

Recall that the integration is performed along the zeroth-order trajectory. The partial of the return function with respect to the initial state x is

$$
\begin{aligned}
P_{i_x}(x, t) = &- \int_t^{t_1-} \frac{\partial R_i}{\partial x} d\tau - R_i(y_1, t_1-)\frac{\partial t_1}{\partial x} \\
&- R_i(y_f, t_f)\frac{\partial t_f}{\partial x} + R_i(y_1, t_1+)\frac{\partial t_1}{\partial x} - \int_{t_1+}^{t_f} \frac{\partial R_i}{\partial x} d\tau
\end{aligned} \tag{C.7}
$$

This equation determines the correction terms to the Lagrange multipliers. Notice that the variation in the time at which the equality constraint is satisfied is explicitly taken into account in the correction terms. Substituting the partial of the expansion of the return function (Eq. (C.1)) into the corner condition of Eq. (C.4) produces

$$\sum_{i=0}^{\infty} \epsilon^i P_{i_x}(y, t_1-) = \sum_{i=0}^{\infty} \epsilon^i P_{i_x}(y, t_1+) + \Pi N_x(y, t_1) \tag{C.8}$$

The solution for the Lagrange multipliers is then determined by collecting the coefficients of like powers of the expansion parameter, ϵ. Therefore, for the zeroth-order term of $i = 0$,

$$P_{0_x}(y, t_1-) = P_{0_x}(y, t_1+) + \Pi N_x(y, t_1)$$

(C.9)

and for higher-order correction terms

$$P_{i_x}(y, t_1-) = P_{i_x}(y, t_1+) \qquad i \geq 1$$

(C.10)

This result implies that the higher-order correction terms for the Lagrange multipliers are continuous across the equality constraint. The jump discontinuity in the Lagrange multipliers is completely taken into account by the zeroth-order term. The higher-order terms of the expansion of the return function are thereby continuous across the corner. The continuity of the Hamiltonian is ensured by substitution of the partial of Eq. (C.1) into Eq. (C.5), which results in

$$\sum_{i=0}^{\infty} \epsilon^i P_{i_x}(y, t_1-) f(y, u, t_1-) = \sum_{i=0}^{\infty} \epsilon^i P_{i_x}(y, t_1+) f(y, u, t_1+)$$

(C.11)

Thus, by collecting terms in like powers of ϵ, the equation

$$P_{i_x}(x, t_1-) f(y, u, t_1-) = P_{i_x}(x, t_1+) f(y, u, t_1+)$$

(C.12)

is obtained. This condition is just the continuity of the expansion terms of the Hamiltonian, i.e., $H_i(y, u, P_x, t_1-) = H_i(y, u, P_x, t_1+)$.

The dynamics of the system are the same after meeting the point constraint as they were before the constraint. Thus, the analytic solution of the states as derived previously is still valid but with a change in the Lagrange multipliers at the dynamic pressure constraint. The relationship between the Lagrange multipliers across the constraint given by Eq. (C.9) can be

used to link the subarc of the zeroth-order trajectory that occurs before the constraint equality is met to the subarc that occurs after the constraint equality is met. These equations are additional conditions that are used to solve the constants associated with the two-point boundary value problem. Since the zeroth-order problem totally accounts for the dynamic pressure equality constraint $N(y, t_1) = 0$, the first-order correction term to the zeroth-order term should be small and the asymptotic expansion should remain valid near the constraint. Therefore according to the method of characteristics, the zeroth-order trajectory can be used as the characteristic curve to determine the higher-order correction terms. In contrast to the unconstrained optimization problem, this zeroth-order trajectory should be close enough to the optimal solution such that only small corrections to the zeroth-order guidance law are necessary.

Appendix D

Analytic Partial Derivatives for Zeroth-Order Solution

The partial derivatives with respect to the general initial state x derived in Chapter 5 for the first-order correction terms are presented here in their explicit form for the initial states, V_0 and γ_0. The equations derived are for the second stage subarc. The initial velocity components expressed in terms of the wind axis states are

$$u_0 = V_0 \cos \gamma_0, \qquad w_0 = -V_0 \sin \gamma_0 \qquad (D.1)$$

and therefore the partial derivatives with respect to the initial velocity and flight path angle are

$$
\begin{aligned}
\frac{\partial u_0}{\partial V_0} &= \cos \gamma_0 \\
\frac{\partial u_0}{\partial \gamma_0} &= -V_0 \sin \gamma_0 \\
\frac{\partial w_0}{\partial V_0} &= -\sin \gamma_0 \\
\frac{\partial w_0}{\partial \gamma_0} &= -V_0 \cos \gamma_0
\end{aligned}
\qquad (D.2)
$$

The partials of the analytic zeroth-order state equations are then expressed as

$$
\begin{aligned}
\frac{\partial u}{\partial V_0} = {} & \cos \gamma_0 - \frac{1}{\sigma \sqrt{a}} \left[\sinh^{-1} \Im_2(m) - \sinh^{-1} \Im_2(m_0) \right] \left(\frac{\partial C_u}{\partial V_0} - \frac{C_u}{2a} \frac{\partial a}{\partial V_0} \right) \\
& - \frac{C_u}{\sigma \sqrt{a}} \left[\frac{1}{\sqrt{1 + \Im_2^2(m)}} \frac{\partial \Im_2}{\partial V_0} - \frac{1}{\sqrt{1 + \Im_2^2(m_0)}} \frac{\partial \Im_2(m_0)}{\partial V_0} \right]
\end{aligned}
\qquad (D.3)
$$

$$\frac{\partial u}{\partial \gamma_0} = -V_0 \sin \gamma_0 - \frac{1}{\sigma \sqrt{a}} \left[\sinh^{-1} \Im_2(m) - \sinh^{-1} \Im_2(m_0) \right] \left(\frac{\partial C_u}{\partial \gamma_0} - \frac{C_u}{2a} \frac{\partial a}{\partial \gamma_0} \right)$$

$$- \frac{C_u}{\sigma \sqrt{a}} \left[\frac{1}{\sqrt{1 + \Im_2^2(m)}} \frac{\partial \Im_2}{\partial \gamma_0} - \frac{1}{\sqrt{1 + \Im_2^2(m_0)}} \frac{\partial \Im_2(m_0)}{\partial \gamma_0} \right] \quad \text{(D.4)}$$

$$\frac{\partial w}{\partial V_0} = -\sin \gamma_0 - \frac{1}{\sigma \sqrt{a}} \left[\sinh^{-1} \Im_2(m) - \sinh^{-1} \Im_2(m_0) \right] \left(\frac{\partial \overline{C}_w}{\partial V_0} - \frac{\overline{C}_w}{2a} \frac{\partial a}{\partial V_0} \right)$$

$$- \frac{\overline{C}_w}{\sigma \sqrt{a}} \left[\frac{1}{\sqrt{1 + \Im_2^2(m)}} \frac{\partial \Im_2}{\partial V_0} - \frac{1}{\sqrt{1 + \Im_2^2(m_0)}} \frac{\partial \Im_2(m_0)}{\partial V_0} \right]$$

$$- \frac{1}{\sigma^2 T \sqrt{c}} \left[\sinh^{-1} \Im_1(m) - \sinh^{-1} \Im_1(m_0) \right] \left(\frac{\partial \lambda_h}{\partial V_0} - \frac{\lambda_h}{2c} \frac{\partial c}{\partial V_0} \right)$$

$$- \frac{\lambda_h}{\sigma^2 T \sqrt{c}} \left[\frac{1}{\sqrt{1 + \Im_1^2(m)}} \frac{\partial \Im_1}{\partial V_0} - \frac{1}{\sqrt{1 + \Im_1^2(m_0)}} \frac{\partial \Im_1(m_0)}{\partial V_0} \right] \quad \text{(D.5)}$$

$$\frac{\partial w}{\partial \gamma_0} = -V_0 \cos \gamma_0 - \frac{1}{\sigma \sqrt{a}} \left[\sinh^{-1} \Im_2(m) - \sinh^{-1} \Im_2(m_0) \right] \left(\frac{\partial \overline{C}_w}{\partial \gamma_0} - \frac{\overline{C}_w}{2a} \frac{\partial a}{\partial \gamma_0} \right)$$

$$- \frac{\overline{C}_w}{\sigma \sqrt{a}} \left[\frac{1}{\sqrt{1 + \Im_2^2(m)}} \frac{\partial \Im_2}{\partial \gamma_0} - \frac{1}{\sqrt{1 + \Im_2^2(m_0)}} \frac{\partial \Im_2(m_0)}{\partial \gamma_0} \right]$$

$$- \frac{1}{\sigma^2 T \sqrt{c}} \left[\sinh^{-1} \Im_1(m) - \sinh^{-1} \Im_1(m_0) \right] \left(\frac{\partial \lambda_h}{\partial \gamma_0} - \frac{\lambda_h}{2c} \frac{\partial c}{\partial \gamma_0} \right)$$

$$- \frac{\lambda_h}{\sigma^2 T \sqrt{c}} \left[\frac{1}{\sqrt{1 + \Im_1^2(m)}} \frac{\partial \Im_1}{\partial \gamma_0} - \frac{1}{\sqrt{1 + \Im_1^2(m_0)}} \frac{\partial \Im_1(m_0)}{\partial \gamma_0} \right] \quad \text{(D.6)}$$

$$\frac{\partial h}{\partial V_0} = -\sin \gamma_0 \frac{(m - m_0)}{\sigma T}$$

$$- \frac{m}{\sigma T \sqrt{a}} \left[\sinh^{-1} \Im_2(m) - \sinh^{-1} \Im_2(m_0) \right] \left(\frac{\partial \overline{C}_w}{\partial V_0} - \frac{\overline{C}_w}{2a} \frac{\partial a}{\partial V_0} \right)$$

$$- m \frac{\overline{C}_w}{\sigma T \sqrt{a}} \left[\frac{1.}{\sqrt{1 + \Im_2^2(m)}} \frac{\partial \Im_2}{\partial V_0} - \frac{1}{\sqrt{1 + \Im_2^2(m_0)}} \frac{\partial \Im_2(m_0)}{\partial V_0} \right]$$

$$+ m \frac{\lambda_h}{2\sigma(\sigma T)^2 C^{3/2}} \left[\sinh^{-1} \Im_1(m) - \sinh^{-1} \Im_1(m_0) \right] \frac{\partial c}{\partial V_0}$$

$$-\frac{m}{\sigma(\sigma T)^2\sqrt{c}}\left[\sinh^{-1}\mathfrak{I}_1(m) - \sinh^{-1}\mathfrak{I}_1(m_0)\right]\frac{\partial\lambda_h}{\partial V_0}$$

$$+\frac{1}{\sigma(\sigma T)^2 c}\left[\sqrt{cm^2 + bm + a} - \sqrt{cm_0^2 + bm_0 + a}\right]\left(\frac{\partial\lambda_h}{\partial V_0} - \frac{\lambda_h}{c}\frac{\partial c}{\partial V_0}\right)$$

$$+\frac{\lambda_h}{2\sigma(\sigma T)^2 c}\left[\frac{m^2\frac{\partial c}{\partial V_0} + m\frac{\partial b}{\partial V_0} + \frac{\partial a}{\partial V_0}}{\sqrt{cm^2 + bm + a}} - \frac{m_0^2\frac{\partial c}{\partial V_0} + m_0\frac{\partial b}{\partial V_0} + \frac{\partial a}{\partial V_0}}{\sqrt{cm_0^2 + bm_0 + a}}\right] \quad \text{(D.7)}$$

$$\frac{\partial h}{\partial\gamma_0} = -\sin\gamma_0\frac{(m - m_0)}{\sigma T}$$

$$-\frac{m}{\sigma T\sqrt{a}}\left[\sinh^{-1}\mathfrak{I}_2(m) - \sinh^{-1}\mathfrak{I}_2(m_0)\right]\left(\frac{\partial\overline{C}_w}{\partial\gamma_0} - \frac{\overline{C}_w}{2a}\frac{\partial a}{\partial\gamma_0}\right)$$

$$-m\frac{\overline{C}_w}{\sigma T\sqrt{a}}\left[\frac{1}{\sqrt{1 + \mathfrak{I}_2^2(m)}}\frac{\partial\mathfrak{I}_2}{\partial\gamma_0} - \frac{1}{\sqrt{1 + \mathfrak{I}_2^2(m_0)}}\frac{\partial\mathfrak{I}_2(m_0)}{\partial\gamma_0}\right]$$

$$+m\frac{\lambda_h}{2\sigma(\sigma T)^2 C^{3/2}}\left[\sinh^{-1}\mathfrak{I}_1(m) - \sinh^{-1}\mathfrak{I}_1(m_0)\right]\frac{\partial c}{\partial\gamma_0}$$

$$-\frac{m}{\sigma(\sigma T)^2\sqrt{c}}\left[\sinh^{-1}\mathfrak{I}_1(m) - \sinh^{-1}\mathfrak{I}_1(m_0)\right]\frac{\partial\lambda_h}{\partial\gamma_0}$$

$$+\frac{1}{\sigma(\sigma T)^2 c}\left[\sqrt{cm^2 + bm + a} - \sqrt{cm_0^2 + bm_0 + a}\right]\left(\frac{\partial\lambda_h}{\partial\gamma_0} - \frac{\lambda_h}{c}\frac{\partial c}{\partial\gamma_0}\right)$$

$$+\frac{\lambda_h}{2\sigma(\sigma T)^2 c}\left[\frac{m^2\frac{\partial c}{\partial\gamma_0} + m\frac{\partial b}{\partial\gamma_0} + \frac{\partial a}{\partial\gamma_0}}{\sqrt{cm^2 + bm + a}} - \frac{m_0^2\frac{\partial c}{\partial\gamma_0} + m_0\frac{\partial b}{\partial\gamma_0} + \frac{\partial a}{\partial\gamma_0}}{\sqrt{cm_0^2 + bm_0 + a}}\right] \quad \text{(D.8)}$$

The partial derivatives of the constants a,b,c, and $\overline{C}_w$ used to express the analytic state equations are

$$\frac{\partial a}{\partial V_0} = 2C_u\frac{\partial C_u}{\partial V_0} + 2\overline{C}_w\frac{\partial\overline{C}_w}{\partial V_0} \quad \text{(D.9)}$$

$$\frac{\partial a}{\partial\gamma_0} = 2C_u\frac{\partial C_u}{\partial\gamma_0} + 2\overline{C}_w\frac{\partial\overline{C}_w}{\partial\gamma_0} \quad \text{(D.10)}$$

$$\frac{\partial b}{\partial V_0} = -\frac{2}{\sigma T}\left[\overline{C}_w\frac{\partial\lambda_h}{\partial V_0} + \lambda_h\frac{\partial\overline{C}_w}{\partial V_0}\right] \quad \text{(D.11)}$$

$$\frac{\partial b}{\partial\gamma_0} = -\frac{2}{\sigma T}\left[\overline{C}_w\frac{\partial\lambda_h}{\partial\gamma_0} + \lambda_h\frac{\partial\overline{C}_w}{\partial\gamma_0}\right] \quad \text{(D.12)}$$

$$\frac{\partial c}{\partial V_0} = \frac{2}{(\sigma T)^2}\frac{\partial \lambda_h}{\partial V_0} \tag{D.13}$$

$$\frac{\partial \overline{C}_w}{\partial V_0} = \frac{\partial C_w}{\partial V_0} + \frac{m_0}{\sigma T}\frac{\partial \lambda_h}{\partial V_0} \tag{D.14}$$

$$\frac{\partial c}{\partial \gamma_0} = \frac{2}{(\sigma T)^2}\frac{\partial \lambda_h}{\partial \gamma_0} \tag{D.15}$$

$$\frac{\partial \overline{C}_w}{\partial V_0} = \frac{\partial C_w}{\partial V_0} + \frac{m_0}{\sigma T}\frac{\partial \lambda_h}{\partial V_0} \tag{D.16}$$

$$\frac{\partial \overline{C}_w}{\partial \gamma_0} = \frac{\partial C_w}{\partial \gamma_0} + \frac{m_0}{\sigma T}\frac{\partial \lambda_h}{\partial \gamma_0} \tag{D.17}$$

Also, the derivatives of the discriminant function $\Delta = 4ac - b^2$ are

$$\frac{\partial \Delta}{\partial V_0} = 4a\frac{\partial c}{\partial V_0} + 4c\frac{\partial a}{\partial V_0} - 2b\frac{\partial b}{\partial V_0} \tag{D.18}$$

$$\frac{\partial \Delta}{\partial \gamma_0} = 4a\frac{\partial c}{\partial \gamma_0} + 4c\frac{\partial a}{\partial \gamma_0} - 2b\frac{\partial b}{\partial \gamma_0} \tag{D.19}$$

The arguments of the inverse hyperbolic sine function are defined in Eq. (5.19) as

$$\Im_1(m) = \frac{2cm + b}{\sqrt{\Delta}} \qquad \Im_2(m) = \frac{2a + bm}{m\sqrt{\Delta}}$$

The resulting partial derivatives of the arguments are

$$\frac{\partial \Im_1}{\partial V_0} = \frac{1}{\sqrt{\Delta}}\left[2(m - \frac{a\Im_1}{\sqrt{\Delta}})\frac{\partial c}{\partial V_0} + (1 + \frac{b\Im_1}{\sqrt{\Delta}})\frac{\partial b}{\partial V_0} - 2c\frac{\Im_1}{\sqrt{\Delta}}\frac{\partial a}{\partial V_0}\right.$$
$$\left. +2c\frac{\partial m}{\partial V_0}\right] \tag{D.20}$$

$$\frac{\partial \Im_1}{\partial \gamma_0} = \frac{1}{\sqrt{\Delta}}\left[2(m - \frac{a\Im_1}{\sqrt{\Delta}})\frac{\partial c}{\partial \gamma_0} + (1 + \frac{b\Im_1}{\sqrt{\Delta}})\frac{\partial b}{\partial \gamma_0} - 2c\frac{\Im_1}{\sqrt{\Delta}}\frac{\partial a}{\partial \gamma_0}\right.$$
$$\left. +2c\frac{\partial m}{\partial \gamma_0}\right] \tag{D.21}$$

$$\frac{\partial \Im_2}{\partial V_0} = \frac{1}{m\sqrt{\Delta}}\left[2(1 - m\frac{c\Im_2}{\sqrt{\Delta}})\frac{\partial a}{\partial V_0} + m(1 + \frac{b\Im_1}{\sqrt{\Delta}})\frac{\partial b}{\partial V_0}\right.$$
$$\left. -ma\frac{\Im_2}{\sqrt{\Delta}}\frac{\partial c}{\partial V_0} - \frac{2}{m}a\frac{\partial m}{\partial V_0}\right] \tag{D.22}$$

$$\frac{\partial \Im_2}{\partial \gamma_0} = \frac{1}{m\sqrt{\Delta}} \left[2(1 - m\frac{c\Im_2}{\sqrt{\Delta}})\frac{\partial a}{\partial \gamma_0} + m(1 + \frac{b\Im_1}{\sqrt{\Delta}})\frac{\partial b}{\partial \gamma_0} \right.$$

$$\left. - ma\frac{\Im_2}{\sqrt{\Delta}}\frac{\partial c}{\partial \gamma_0} - \frac{2}{m}a\frac{\partial m}{\partial \gamma_0} \right] \tag{D.23}$$

$$\tag{D.24}$$

Note that unless the partial deriviatives are evaluated at the terminal manifold the partials $\frac{\partial m}{\partial V_0}$ and $\frac{\partial m}{\partial \gamma_0}$ are zero. Using the partial derivative chain rule for a trignometric function, the partials of the inverse hyperbolic sine function are obtained.

$$\frac{\partial}{\partial V_0}\left(\sinh^{-1}\Im_1\right) = \frac{1}{\sqrt{1 + \Im_1^2}}\frac{\partial \Im_1}{\partial V_0} \tag{D.25}$$

$$\frac{\partial}{\partial \gamma_0}\left(\sinh^{-1}\Im_2\right) = \frac{1}{\sqrt{1 + \Im_2^2}}\frac{\partial \Im_2}{\partial \gamma_0} \tag{D.26}$$

Therefore, all the partials derivatives needed to evaluate the partial derivatives of the analytic state equations along the zeroth-order trajectory depend on the eight constant partial derivatives $\frac{\partial \lambda_h}{\partial V_0}, \frac{\partial C_u}{\partial V_0}, \frac{\partial C_w}{\partial V_0}, \frac{\partial m_f}{\partial V_0}, \frac{\partial \lambda_h}{\partial \gamma_0}, \frac{\partial C_u}{\partial \gamma_0}, \frac{\partial C_u}{\partial \gamma_0}, \frac{\partial m_f}{\partial \gamma_0}$. These partial derivatives are functions of the solution to the two point-boundary value problem. Therefore, they are constant when integrating the forcing function R_1 from the initial to the final conditions but they change as new initial conditions are given when the guidance scheme is used in feedback.

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13. ABSTRACT (Maximum 200 words)

A real-time guidance scheme for the problem of maximizing the payload into orbit subject to the equations of motion for a rocket over a spherical, nonrotating Earth is presented. An approximate optimal launch guidance law is developed based upon an asymptotic expansion of the Hamilton-Jacobi-Bellman or dynamic programming equation. The expansion is performed in terms of a small parameter, which is used to separate the dynamics of the problem into primary and perturbation dynamics. For the zeroth-order problem the small parameter is set to zero and a closed-form solution to the zeroth-order expansion term of Hamilton-Jacobi-Bellman equation is obtained. Higher-order terms of the expansion include the effects of the neglected perturbation dynamics. These higher-order terms are determined from the solution of first-order linear partial differential equations requiring only the evaluation of quadratures. This technique is preferred as a real-time on-line guidance scheme to alternative numerical iterative optimization schemes because of the unreliable convergence properties of these iterative guidance schemes and because the quadratures needed for the approximate optimal guidance law can be performed rapidly and by parallel processing. Even if the approximate solution is not nearly optimal, when using this technique the zeroth-order solution always provides a path which satisfies the terminal constraints. Results for two-degree-of-freedom simulations are presented for the simplified problem of flight in the equatorial plane and compared to the guidance scheme generated by the shooting method which is an iterative second-order technique.

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